

# Vertex operators in solvable lattice models

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## Abstract

We formulate the basic properties of  $q$ -vertex operators in the context of the Andrews-Baxter-Forrester (ABF) series, as an example of face-interaction models, derive the  $q$ -difference equations satisfied by their correlation functions, and establish their connection with representation theory. We also discuss the  $q$ -difference equations of the Kashiwara-Miwa (KM) series, as an example of edge-interaction models.

Next, the Ising model—the simplest special case of both ABF and KM series—is studied in more detail using the Jordan-Wigner fermions. In particular, all matrix elements of vertex operators are calculated.

## 1 Introduction

In this paper, we continue our study of the role of vertex operators and difference equations in computing correlation functions in off-critical solvable lattice models. We start with a review of the conformal field theory (CFT) approach to computing critical correlation functions in WZW-type models: models with an affine Lie algebra symmetry, then compare it with our approach in solvable lattice models of the vertex type.

### At criticality

In CFT, the correlation functions on the projective plane  $\mathbf{P}^1$ :

$$\langle \lambda_0 | \Phi_{\lambda_1}^{\lambda_0, V^1}(\zeta_1) \cdots \Phi_{\lambda_n}^{\lambda_{n-1}, V^n}(\zeta_n) | \lambda_n \rangle \quad (1)$$

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are given in terms of vertex operators acting as intertwiners in the following sense:

$$\Phi_{\lambda_j}^{\lambda_{j-1}, V^j}(\zeta) : V(\lambda_j) \rightarrow V(\lambda_{j-1}) \otimes V_{\zeta}^j \quad (2)$$

where  $V(\lambda)$  is a highest weight representation with highest weight  $\lambda$  and highest weight vector  $|\lambda\rangle$ , and  $V_{\zeta}^j$  is a finite dimensional representation with a spectral parameter  $\zeta$ . For simplicity, we take all  $V^j$ 's to be isomorphic to the same finite dimensional representation  $V$ .

The correlation functions satisfy the KZ equation, a system of differential equations with respect to the variables  $\zeta_1, \dots, \zeta_n$ . The solutions to the KZ equation have a branch point at  $\zeta_j = \zeta_k$ , and their monodromy is dictated by the commutation relations of the vertex operators

$$\Phi_{\lambda_4}^{\lambda_1, V^1}(\zeta_1) \Phi_{\lambda_3}^{\lambda_4, V^2}(\zeta_2) = \sum_{\lambda_2} \Phi_{\lambda_2}^{\lambda_1, V^2}(\zeta_2) \Phi_{\lambda_3}^{\lambda_2, V^1}(\zeta_1) W \begin{pmatrix} \lambda_1 & \lambda_4 \\ \lambda_2 & \lambda_3 \end{pmatrix}, \quad (3)$$

which is understood in terms of an appropriate analytic continuation, and the connection coefficients  $W$  are constants.

## Off criticality

Away from criticality, one expects appropriate generalizations, or deformations of the structures that appear at criticality. In [12],  $q$ -deformations of the vertex operators and the KZ equation were obtained. There is a number of major differences between the critical, or classical (i.e.  $q = 1$ ) and  $q$ -deformed cases. To start with, the  $q$ -deformed KZ equation is a system of difference rather than differential equations. Furthermore, the commutation relation changes to

$$\begin{aligned} & R^{V^1, V^2}(\zeta_1/\zeta_2) \Phi_{\lambda_4}^{\lambda_1, V^1}(\zeta_1) \Phi_{\lambda_3}^{\lambda_4, V^2}(\zeta_2) \\ &= \sum_{\lambda_2} \Phi_{\lambda_2}^{\lambda_1, V^2}(\zeta_2) \Phi_{\lambda_3}^{\lambda_2, V^1}(\zeta_1) W \left( \begin{array}{cc|c} \lambda_1 & \lambda_4 & \zeta_1/\zeta_2 \\ \lambda_2 & \lambda_3 & \end{array} \right). \end{aligned} \quad (4)$$

thus, the  $R$ -matrix appears on the left hand side, and the connection coefficients  $W$  now depend on the spectral parameters. In the  $q$ -deformed case, the multivaluedness reduces to an infinite set of poles at  $\zeta_j/\zeta_k = q^n$  for certain integers  $n \neq 0$ .

Another crucial difference is that, in the  $q$ -deformed case, it *matters* whether the finite dimensional part  $V_{\zeta}^j$  appears as the right or left component in the action of the vertex operator: there are *two* different vertex operators, depending on the position of  $V_{\zeta}^j$  in the action. In [9, 14], they are distinguished as type I ( $V_{\zeta}^j$  is to the right) and type II ( $V_{\zeta}^j$  is to the left) vertex operator. In this language, (2) shows the action of a type I vertex operator. The operators that

appear in (4) are also of type I. They are used to compute the correlation functions in the form

$$\mathrm{tr}_{V(\lambda_0)} q^{-2\rho} \Phi_{\lambda_1}^{\lambda_0, V^1}(\zeta_1) \cdots \Phi_{\lambda_n}^{\lambda_{n-1}, V^n}(\zeta_n), \quad (5)$$

where  $\rho$  is a certain grading operator, and  $\lambda_0 = \lambda_n$ . The above form of the correlation functions suggests a connection to conformal field theory defined on the annulus. Recently, an interesting paper appeared in this direction [10], where the classical and twisted versions of the trace function were found to satisfy the classical and elliptic versions of the KZ equation, respectively. On the other hand, the Type II vertex operators serve as creation and annihilation operators that diagonalize the Hamiltonian.

## Previous results

In our previous work on the off-critical correlation functions we started with the XXZ spin chain, then considered the 8-vertex model. In the case of the XXZ spin chain, we followed the representation theoretic approach, since that was possible. The trace correlations that we obtained, as in (5), also satisfy a version of the  $q$ -deformed KZ equation (see (5) in [15]) although the derivation in this case is different from that in the case of correlations that are matrix elements as in (1): it relies on the cyclic property of the trace and the following commutation relations of the vertex operators, which are valid in the principal grading for  $V_z$ :

$$\zeta^\rho \Phi_{\lambda'}^{\lambda, V}(\zeta') = \Phi_{\lambda'}^{\lambda, V}(\zeta'/\zeta) \zeta^\rho, \quad (6)$$

$$R^{V^1, V^2}(\zeta_1/\zeta_2) \Phi_{\lambda_2}^{\lambda_1, V^1}(\zeta_1) \Phi_{\lambda_3}^{\lambda_2, V^2}(\zeta_2) = \Phi_{\lambda_2}^{\lambda_1, V^2}(\zeta_2) \Phi_{\lambda_3}^{\lambda_2, V^1}(\zeta_1). \quad (7)$$

Compare (4) with (7): The reason that the summation on the right hand side disappears is that the level of the highest weights and the finite dimensional representation  $V$  are such that that the highest weight  $\lambda$  in

$$\Phi_{\lambda'}^{\lambda, V}(\zeta) : V(\lambda') \rightarrow V(\lambda) \otimes V_\zeta, \quad (8)$$

is uniquely determined from  $\lambda'$ . Specifically, in the case of the XXZ spin chain, we have: level = 1, and  $\dim V = 2$  for  $\widehat{sl}(2)$ ; refer to the definition of *perfect crystals* in [18]. This trivializes the connection coefficients  $W$ , that would otherwise appear on the right hand side.

The above simplification was used in the derivation of the  $q$ -difference equations in the 8-vertex model (see (6) in [15]). In [15], an argument without explicit use of representation theory, for deriving similar equations in the context of the 8-vertex model was proposed. The idea in this case is an extension of Baxter's corner transfer matrix (CTM). We replace  $V(\lambda)$  with the space  $\mathcal{H}^{(i)}$  spanned by the eigenvectors of the CTM with the specific, say  $i$ -th boundary condition,  $q$  is regarded as the crossing parameter, and  $\zeta^{-\rho}$  as the CTM. The

basic idea is to interpret the vertex operator as a graphical insertion of a half line. By doing that, the commutation relations (6) and (7) follow from a simple graphical argument, and as a corollary the elliptic difference equation is shown to be valid for the correlation functions of the eight-vertex model. A mathematical setting for the eight-vertex model, in which the vertex operators are given a mathematical basis, in terms of e.g. representation theory, is still unavailable.

## Outline of results

Now, we turn to the subject and results of the present paper. We wish to pursue both the graphical and the representation theory approaches to the vertex operators and difference equations, but this time in the context of the ABF models.

In Section 1, we go through the graphical procedure followed in the case of the 8-vertex model, in order to obtain the vertex operators and the  $q$ -difference equations of the ABF models. The commutation relation reads as

$$\begin{aligned} & \Phi^{(i,i+1)}(\zeta_1)_{l_1,l_2} \Phi^{(i+1,i)}(\zeta_2)_{l_2,l_3} = \\ & \sum_{l_4} W \left( \begin{array}{cc} l_1 & l_4 \\ l_2 & l_3 \end{array} \middle| \zeta_1/\zeta_2 \right) \Phi^{(i,i+1)}(\zeta_2)_{l_1,l_4} \Phi^{(i+1,i)}(\zeta_1)_{l_4,l_3}, \end{aligned} \quad (9)$$

where the indices  $i$  and  $l$  label the different sectors in the space of states, and the heights on the lattice, respectively as will be explained later. Next, using the results of [16], we give a representation theoretic realization of the vertex operators satisfying the above commutation relations.

At this point, let us emphasize the following: The graphical derivation of the difference equations applies to other solvable models, even when the representation theoretic setting is not available. Examples are edge interaction models, such as the Kashiwara-Miwa and Chiral Potts models. We give a brief derivation of the difference equations of the Kashiwara-Miwa model. The treatment is basically a special case of that of the ABF models.

In the case of the XXZ spin chain, we have, not only the representation theoretic realization of the vertex operators, but also a more powerful realization in terms of free bosons. In [14], an integral representation was obtained for the correlation functions using bosonization. In the ABF case, we do not know of a bosonization, or any other tool, that would provide us with an integral formula for the correlation functions. However, in the case of the Ising model, which is an intersection of all the models discussed above, with the exception of the XXZ spin chain, an alternative tool is available: the model can be formulated in terms of free fermions.

The diagonalization of the Ising CTM was given in [2] and [8] using Jordan-Wigner fermions. In this case, the role of type II vertex operators is basically played by the diagonalized fermions. In Section 2, we use this formalism to

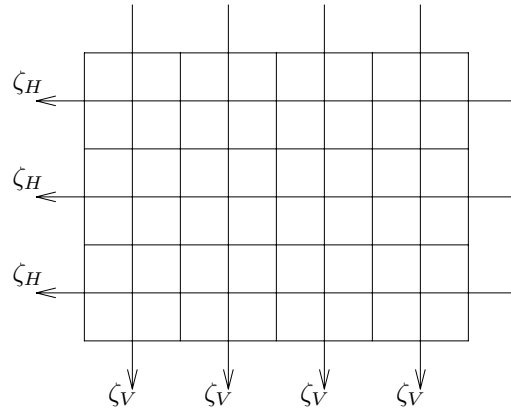
obtain all the matrix elements of vertex operators, and solve the difference equation for the simplest case to obtain the two point trace function.

## 2 The ABF models

In this section, after reviewing the corner transfer matrix (CTM) method, we introduce the vertex operators (VO's) of the ABF models. We deduce the commutation relations among them, express the correlation functions as traces of VO's, and derive  $q$ -difference equations for the traces. The arguments rely on the assumption that the relevant operators are well defined in the infinite lattice limit. In the last subsection we relate this heuristic construction to the representation theoretical formulation proposed earlier in [16], showing that the latter provides a mathematical model for the VO's satisfying all the expected properties.

### Definitions

We consider the ABF models on a square lattice [1], and associate to each site  $j$  a local state variable  $\sigma_j \in l_j = 1, \dots, L - 1$ , where  $L$  is a positive integer ( $\geq 4$ ). Next, we draw oriented lines on the dual lattice, and associate a spectral parameter  $\zeta_H$  with each horizontal line and  $\zeta_V$  with each vertical line:



To each configuration of local variables  $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$  surrounding a face, we associate a Boltzmann weight. The latter will be non-vanishing only if the configuration is admissible: if the local sites  $j$  and  $j'$  are connected by an edge then  $\sigma_j - \sigma_{j'} = \pm 1$ . The Boltzmann weights depend on  $\zeta = \zeta_V / \zeta_H$ , and also on a parameter  $x$  such that  $0 < x < 1$ , which can be thought of as temperature variable:  $x \rightarrow 0$  corresponds to the low temperature limit;  $x \rightarrow 1$  corresponds to the critical limit. We shall restrict our attention to the region

$$0 < x < \zeta < 1$$

which corresponds to regime III in [1]. Since  $x$  will remain constant, we will display only the dependence on  $\zeta$  and suppress the dependence on  $x$ ; for instance, the CTM below will be written as  $A_4^{(i)}(\zeta)$ .

Set  $p = x^{2L}$ , and define

$$\begin{aligned}\Theta_p(z) &= (z; p)_\infty (p/z; p)_\infty (p; p)_\infty, \\ (z; q_1, \dots, q_k)_\infty &= \prod_{n_1, \dots, n_k=0}^{\infty} (1 - z q_1^{n_1} \dots q_k^{n_k}).\end{aligned}$$

With these notations, the Boltzmann weights for the admissible configurations are

$$\begin{array}{c} \sigma_4 = l_4 \quad \sigma_3 = l_3 \\ \leftarrow \zeta_H \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad \rightarrow = W \left( \begin{array}{cc} l_4 & l_3 \\ l_1 & l_2 \end{array} \middle| \zeta \right) \quad (\zeta = \zeta_V / \zeta_H) \\ \sigma_1 = l_1 \quad \sigma_2 = l_2 \\ \downarrow \zeta_V \end{array}$$

$$W \left( \begin{array}{cc} l & l \pm 1 \\ l \pm 1 & l \pm 2 \end{array} \middle| \zeta \right) = \frac{1}{\kappa(\zeta)} \zeta \frac{\Theta_p(x^2 \zeta^{-2})}{\Theta_p(x^2)}, \quad (10)$$

$$W \left( \begin{array}{cc} l & l \pm 1 \\ l \pm 1 & l \end{array} \middle| \zeta \right) = \frac{1}{\kappa(\zeta)} \frac{\Theta_p(x^{2l} \zeta^{\pm 2})}{\Theta_p(x^{2l})}, \quad (11)$$

$$W \left( \begin{array}{cc} l & l \pm 1 \\ l \mp 1 & l \end{array} \middle| \zeta \right) = \frac{1}{\kappa(\zeta)} \frac{x}{\zeta} \frac{\Theta_p(\zeta^2) \Theta_p(x^{2(l \pm 1)})}{\Theta_p(x^2) \Theta_p(x^{2l})}. \quad (12)$$

Here the normalizing factor  $\kappa(\zeta)$  is so chosen that the partition function per site is equal to 1. From the standard inversion trick we find

$$\kappa(\zeta) = \bar{\kappa}(\zeta) \frac{\Theta_p(x^2 \zeta^{-2})}{\Theta_p(x^2)}, \quad (13)$$

$$\bar{\kappa}(\zeta) = \frac{(x^2 \zeta^2; p, x^4)_\infty (p x^2 \zeta^2; p, x^4)_\infty (x^4 \zeta^{-2}; p, x^4)_\infty (p \zeta^{-2}; p, x^4)_\infty}{(x^2 \zeta^{-2}; p, x^4)_\infty (p x^2 \zeta^{-2}; p, x^4)_\infty (x^4 \zeta^2; p, x^4)_\infty (p \zeta^2; p, x^4)_\infty} \quad (14)$$

Note that  $\kappa(\zeta) = \kappa(x \zeta^{-1})$ . For later use we set

$$g_l = \Theta_p(x^{2l}). \quad (15)$$

Instead of taking the  $\zeta_H, \zeta_V$  to be all the same, we could equally well assign spectral parameters that can vary independently from line to line. The resulting

model is  $Z$ -invariant in Baxter's sense. Later we will take advantage of this freedom to vary some of the spectral parameters.

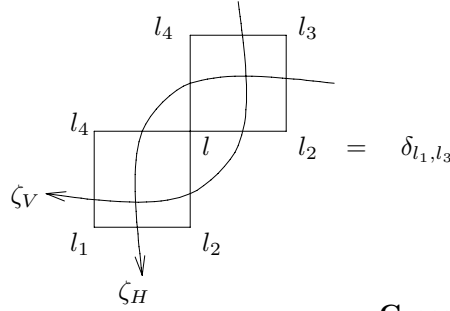
The basic properties of the Boltzmann weights are as follows:

**Initial condition**

$$W \left( \begin{array}{cc|c} l_4 & l_3 & 1 \\ l_1 & l_2 & \end{array} \right) = \delta_{l_1, l_3}. \quad (16)$$

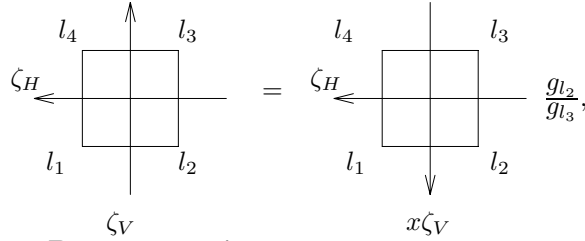
**Unitarity relation**

$$\sum_l W \left( \begin{array}{cc|c} l_4 & l & \zeta^{-1} \\ l_1 & l_2 & \end{array} \right) W \left( \begin{array}{cc|c} l_4 & l_3 & \zeta \\ l & l_2 & \end{array} \right) = \delta_{l_1, l_3}. \quad (17)$$



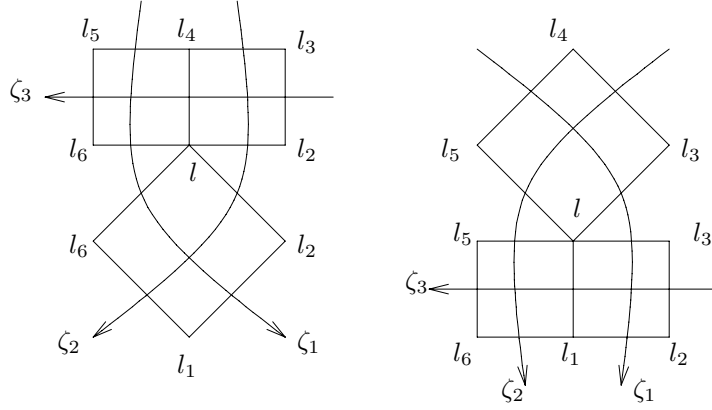
**Crossing symmetry**

$$W \left( \begin{array}{cc|c} l_3 & l_2 & \zeta^{-1} \\ l_4 & l_1 & \end{array} \right) = W \left( \begin{array}{cc|c} l_4 & l_3 & x\zeta \\ l_1 & l_2 & \end{array} \right) \frac{g_{l_2}}{g_{l_3}} \quad (18)$$



**Yang-Baxter equation**

$$\begin{aligned} & \sum_l W \left( \begin{array}{cc|c} l_4 & l_3 & \zeta_2/\zeta_3 \\ l & l_2 & \end{array} \right) W \left( \begin{array}{cc|c} l_5 & l_4 & \zeta_1/\zeta_3 \\ l_6 & l & \end{array} \right) W \left( \begin{array}{cc|c} l_6 & l & \zeta_1/\zeta_2 \\ l_1 & l_2 & \end{array} \right) \\ &= \sum_l W \left( \begin{array}{cc|c} l_5 & l_4 & \zeta_1/\zeta_2 \\ l & l_3 & \end{array} \right) W \left( \begin{array}{cc|c} l & l_3 & \zeta_1/\zeta_3 \\ l_1 & l_2 & \end{array} \right) W \left( \begin{array}{cc|c} l_5 & l & \zeta_2/\zeta_3 \\ l_6 & l_1 & \end{array} \right) \end{aligned} \quad (19)$$



### Corner transfer matrices

Let us label the vertices on the lattice as  $(j_1, j_2)$  ( $j_1, j_2 \in \mathbf{Z}$ ) so that  $j_1$  is increasing to the left and  $j_2$  is increasing in the upward direction. For  $i = 0, 1 \in \mathbf{Z}/2\mathbf{Z}$  and  $m = 1, \dots, L-2$ , we denote by  $C_{m,m+1}^{(i)}$  the ground state configuration such that

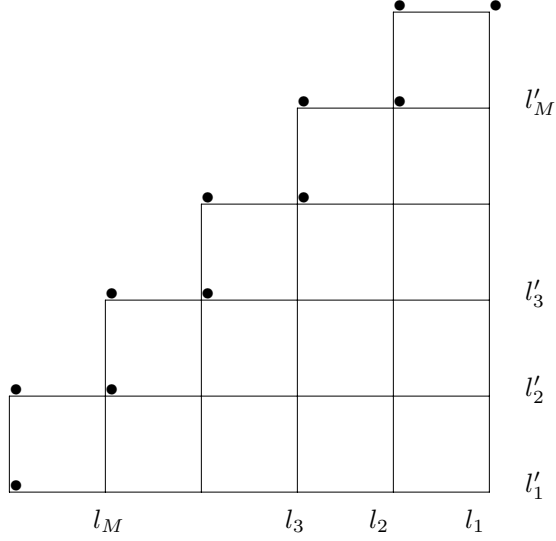
$$\sigma_{(j_1, j_2)} = m + 1 \text{ if } j_1 + j_2 = i + 1 \bmod 2, \quad (20)$$

$$= m \text{ if } j_1 + j_2 = i \bmod 2. \quad (21)$$

In the following discussion we choose and fix  $m$ , and suppress it in our notation.

Consider the NW-quadrant of the lattice consisting of  $M+1$  rows and  $M+1$  columns, with the vertex  $(1, 1)$  at the SE corner:





On the boundary shown by bullets, the variables are fixed to have the same value as in  $C_{m,m+1}^{(i)}$ . Let us choose a configuration of the local variables  $\mathbf{l}' = (l'_1, \dots, l'_M)$  (resp.  $\mathbf{l} = (l_1, \dots, l_M)$ ) on the vertical (resp. horizontal) half line  $(1, 1), \dots, (1, M)$  (resp.  $(1, 1), \dots, (M, 1)$ ). Calculate the partition function of this quadrant while fixing these variables. The  $(\mathbf{l}, \mathbf{l}')$  matrix element of the CTM  $A_4^{(i)}(\zeta)$  is defined to be this partition function. Here  $i$  refers to the choice of the boundary configuration. The matrix element is defined to be zero unless  $l_1 = l'_1$ . It is known that in the limit  $M \rightarrow \infty$  the CTM takes the simple form

$$A_4^{(i)}(\zeta) \sim \zeta^{D^{(i)}},$$

where  $D^{(i)}$  is the CTM Hamiltonian: a  $\zeta$ -independent operator whose spectrum is contained in the set  $\{0, 1, 2, \dots\}$ . The symbol  $\sim$  indicates equality up to multiplication by a scalar. Such a scalar factor is irrelevant in the computation of the correlation functions.

Set

$$\begin{aligned} \bar{l}_k^{(i)} &= m \text{ if } k = i + 1 \bmod 2, \\ &= m + 1 \text{ if } k = i \bmod 2. \end{aligned}$$

Consider the vector space  $\mathcal{H}^{(i)}$  whose basis elements are admissible configurations  $\mathbf{l} = (l_1, l_2, \dots)$  satisfying the boundary condition

$$l_k = \bar{l}_k^{(i)} \text{ if } k \gg 1.$$

Let us write  $|\mathbf{l}\rangle$  when we wish to emphasize that it is a vector in this space. Formally the CTM  $A_4^{(i)}(\zeta)$  or  $D^{(i)}$  act on  $\mathcal{H}^{(i)}$ , and their eigenvectors are certain

infinite linear combinations of the  $|\mathbf{l}\rangle$ . In our heuristic approach, we identify  $\mathcal{H}^{(i)}$  with the vector space spanned by these eigenvectors.

We denote by  $S_1^{(i)}, G_1^{(i)}$  the operators

$$S_1^{(i)}|\mathbf{l}\rangle = l_1|\mathbf{l}\rangle, \quad G_1^{(i)}|\mathbf{l}\rangle = g_{l_1}|\mathbf{l}\rangle, \quad (22)$$

where  $g_l$  is defined in (15). By construction  $D^{(i)}$  commutes with  $S_1^{(i)}$ . Hence each eigenspace  $\mathcal{H}_l^{(i)}$  of  $S_1^{(i)}$  with eigenvalue  $l$  is invariant under the action of  $D^{(i)}$ , and we have  $\mathcal{H}^{(i)} = \oplus_l \mathcal{H}_l^{(i)}$ .

The CTMs for the NE, SE and SW quadrants  $A_1^{(i)}(\zeta), A_2^{(i)}(\zeta), A_3^{(i)}(\zeta)$  are defined similarly. Let  $R^{(i)}$  be the diagonal matrix acting on  $\mathcal{H}^{(i)}$

$$R^{(i)}|\mathbf{l}\rangle = \prod_{k=1}^{\infty} \frac{g_{l_k}}{g_{\tilde{l}_k^{(i)}}}|\mathbf{l}\rangle. \quad (23)$$

Then the crossing symmetry (18) implies that

$$A_3^{(i)}(\zeta) \sim R^{(i)} \cdot (x/\zeta)^{D^{(i)}}, \quad (24)$$

$$A_2^{(i)}(\zeta) \sim R^{(i)} \cdot \zeta^{D^{(i)}} \cdot R^{(i)-1}, \quad (25)$$

$$A_1^{(i)}(\zeta) \sim G_1^{(i)} \cdot (x/\zeta)^{D^{(i)}} \cdot R^{(i)-1}. \quad (26)$$

Therefore, we have

$$A_1^{(i)}(\zeta)A_2^{(i)}(\zeta)A_3^{(i)}(\zeta)A_4^{(i)}(\zeta) \sim G_1^{(i)} \cdot x^{2D^{(i)}}. \quad (27)$$

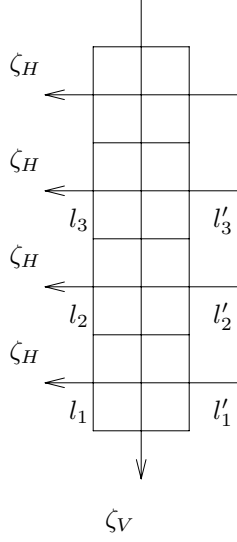
## Vertex operators

To be able to write down expressions for the correlation functions, we need to introduce the VO's

$$\Phi^{(i+1,i)}(\zeta) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(i+1)}.$$

Notice the difference between the vertex operators introduced above, and those defined in the previous section. The point is that the VO's discussed in the previous section are defined in the context of vertex models. The VO's that we discuss in this section are defined in the context of face models. We will refer to the former as VO's of the *vertex type* and to the latter as VO's of the *face type*. The relation between them will be discussed further below, in the subsection "Construction by representation theory".

Graphically the  $(\mathbf{l}, \mathbf{l}')$  matrix element of  $\Phi^{(i+1,i)}(\zeta)$  is defined to be the product of Boltzmann weights as follows



The CTM's, their Hamiltonians, and the vertex operators act on specific subspaces of the full space of states. The action on a certain subspace  $\mathcal{H}_l^{(i)}$  will be indicated by the subscripts  $l, l'$ , as follows:

$$\begin{aligned} A_k^{(i)}(\zeta)_l, D_l^{(i)} &: \mathcal{H}_l^{(i)} \rightarrow \mathcal{H}_l^{(i)}, \\ \Phi^{(i,i')}(\zeta)_{l,l'} &: \mathcal{H}_{l'}^{(i')} \rightarrow \mathcal{H}_l^{(i)}. \end{aligned}$$

We shall argue that, with an appropriate choice of a scalar factor, the VO's enjoy the following basic properties:

#### Homogeneity

$$\zeta^{D^{(i+1)}} \circ \Phi^{(i+1,i)}(\zeta') \circ \zeta^{-D^{(i)}} = \Phi^{(i+1,i)}(\zeta'/\zeta). \quad (28)$$

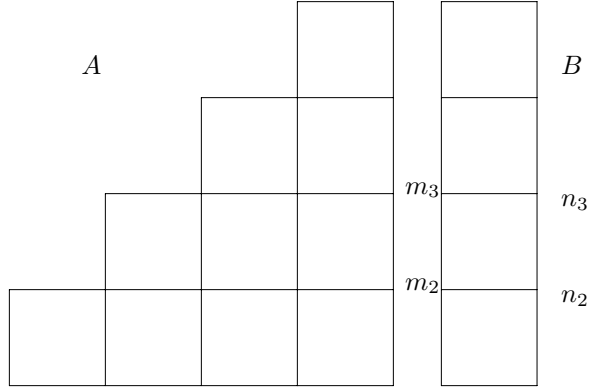
#### Commutation relations

$$\begin{aligned} \sum_{l_3} W \left( \begin{array}{cc} l_4 & l_3 \\ l_1 & l_2 \end{array} \middle| \zeta_1/\zeta_2 \right) \Phi^{(i,i+1)}(\zeta_1)_{l_4,l_3} \Phi^{(i+1,i)}(\zeta_2)_{l_3,l_2} \\ = \Phi^{(i,i+1)}(\zeta_2)_{l_4,l_1} \Phi^{(i+1,i)}(\zeta_1)_{l_1,l_2}. \end{aligned} \quad (29)$$

#### Normalization

$$\sum_{l_2} g_{l_2} \Phi^{(i,i+1)}(x\zeta)_{l_1,l_2} \Phi^{(i+1,i)}(\zeta)_{l_2,l_1} = id_{\mathcal{H}_{l_1}^{(i)}}. \quad (30)$$

To deduce (28), consider the following:

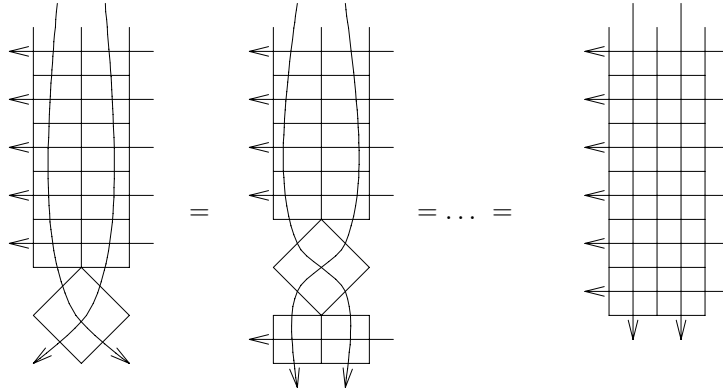


$$l_4 = p_3 \quad l_3 = p_2 \quad l_2 = m_1 = p_1 \quad l_1 = n_1$$

The figure shows the composition of  $A_4^{(i+1)}(\zeta)$  and  $\Phi^{(i+1,i)}(\zeta)$ . The  $(\mathbf{p}, \mathbf{m})$ -component of  $A_4^{(i+1)}(\zeta)$  is represented by the part A and the  $(\mathbf{m}, \mathbf{n})$ -component of  $\Phi^{(i+1,i)}(\zeta)$  is represented by the part B. The composition of A followed by B (with  $(m_2, m_3, \dots)$  summed over) can be viewed as the  $(\mathbf{l}, \mathbf{n})$ -component of  $A_4^{(i)}(\zeta)$ , except that the product  $A_4^{(i+1)}(\zeta)\Phi^{(i+1,i)}(\zeta)$  is acting as  $\mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i+1)}$  while  $A_4^{(i)}(\zeta)$  as  $\mathcal{H}^{(i)} \rightarrow \mathcal{H}^{(i)}$ . To obtain an equality,  $\mathcal{H}^{(i)}$  and  $\mathcal{H}^{(i+1)}$  must be intertwined. In view of the initial condition (16) for the Boltzmann weights, the required intertwining action is exactly what  $\Phi^{(i+1,i)}(1)$  does. Therefore, we have

$$A_4^{(i+1)}(\zeta)_{l_2} \Phi^{(i+1,i)}(\zeta)_{l_2, l_1} = \Phi^{(i+1,i)}(1)_{l_2, l_1} A_4^{(i)}(\zeta)_{l_1}.$$

The normalization (30) can be shown using unitarity (17) and crossing symmetry (18). To motivate (29) we make use of the Yang-Baxter equation.



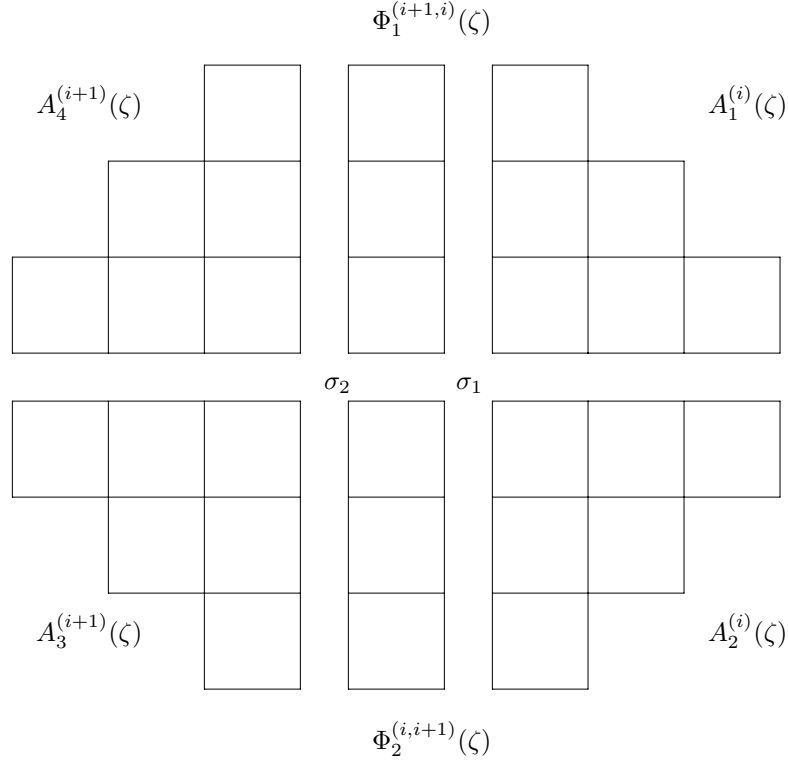
In the above,

the face added in the first figure is pushed up to infinity, using the YB equations, and because of the normalization by  $\kappa$  its *finite* effect will disappear in the *infinite* limit.

Now let us relate the VO's to the correlation functions. As an illustration we discuss the simplest situation. Look at the figure below. We wish to calculate the local probability of the adjacent variables  $\sigma_1$  and  $\sigma_2$  taking the values  $l_1$  and  $l_2$ , respectively. We denote this probability by  $P^{(i)}(l_1, l_2)$ .

Divide the whole lattice into 6 pieces. The main parts are the CTM's. The remaining pieces give the graphical definition of two types of VO's

$$\Phi_1^{(i+1,i)}(\zeta) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(i+1)} \quad \Phi_2^{(i,i+1)}(\zeta) : \mathcal{H}^{(i+1)} \longrightarrow \mathcal{H}^{(i)}.$$



The operator  $\Phi_1^{(i+1,i)}(\zeta)$  is the same as  $\Phi^{(i+1,i)}(\zeta)$  that we introduced. In much the same way as for the CTM (24)-(26),  $\Phi_2^{(i,i+1)}(\zeta)$  can be written in terms of  $\Phi^{(i+1,i)}(\zeta)$ . Namely using the crossing symmetry (18) we have

$$\Phi_2^{(i,i+1)}(\zeta) \sim R^{(i)} \Phi^{(i,i+1)}(\zeta) G_1^{(i+1)} R^{(i+1)-1}, \quad (31)$$

where  $G_1^{(i)}$  and  $R^{(i)}$  are defined in (22) and (23).

Using these operators, we have

$$P^{(i)}(l_1, l_2) = \frac{Q^{(i)}(l_1, l_2)}{\sum_{l'_1, l'_2} Q^{(i)}(l'_1, l'_2)}, \quad (32)$$

$$Q^{(i)}(l_1, l_2) = \text{tr}_{\mathcal{H}_{l_1}^{(i)}} \left( A_1^{(i)}(\zeta)_{l_1} A_2^{(i)}(\zeta)_{l_1} \times \right. \\ \left. \Phi_2^{(i, i+1)}(\zeta)_{l_1, l_2} A_3^{(i+1)}(\zeta)_{l_2} A_4^{(i+1)}(\zeta)_{l_2} \Phi_1^{(i+1, i)}(\zeta)_{l_2, l_1} \right). \quad (33)$$

By (28) and (31) we can change the order of the product

$$\Phi_2^{(i, i+1)}(\zeta)_{l_1, l_2} A_3^{(i+1)}(\zeta)_{l_2} A_4^{(i+1)}(\zeta)_{l_2}$$

in (33), to find

$$Q^{(i)}(l_1, l_2) = g_{l_1} g_{l_2} \times \\ \text{tr}_{\mathcal{H}_{l_1}^{(i)}} \left( x^{2D_{l_1}^{(i)}} \Phi^{(i, i+1)}(x\zeta)_{l_1, l_2} \Phi^{(i+1, i)}(\zeta)_{l_2, l_1} \right). \quad (34)$$

If we choose the normalization (30) then the denominator  $\sum_{l_1, l_2} Q^{(i)}(l_1, l_2)$  in (32) is equal to  $\text{tr}_{\mathcal{H}^{(i)}} \left( G_1^{(i)} x^{2D^{(i)}} \right)$ . In a similar way, it is straightforward to relate the general correlators to traces of products of VO's.

### The $q$ -difference equation

Suppose that  $n$  is even and the one-dimensional configuration  $[l_1, l_2, \dots, l_n, l_{n+1}]$  is admissible:  $l_{k+1} = l_k \pm 1$ , and  $l_{n+1} = l_1$ . We call such a configuration a cyclic path of length  $n$  and denote it by  $[l_1, \dots, l_n]$ . Consider the vector space  $\mathcal{V}_n$  having the cyclic paths of length  $n$  as basis. If  $l_1, \dots, l_n$  do not satisfy the above admissibility condition we regard  $[l_1, \dots, l_n]$  to be a null vector. On  $\mathcal{V}_n$  we define an operator  $W_k(\zeta)$  ( $2 \leq k \leq n$ ) by

$$W_k(\zeta)[l_1, \dots, l_n] = \sum_{l'_k} W \left( \begin{matrix} l_{k-1} & l_k \\ l'_k & l_{k+1} \end{matrix} \middle| \zeta \right) [l_1, \dots, l_{k-1}, l'_k, l_{k+1}, \dots, l_n]. \quad (35)$$

Define further

$$F_{l_1, \dots, l_n}^{(i)}(\zeta_1, \dots, \zeta_n) = \text{tr}_{\mathcal{H}_{l_1}^{(i)}} \left( x^{2D_{l_1}^{(i)}} \Phi^{(i, i+1)}(\zeta_1)_{l_1, l_2} \dots \Phi^{(i+1, i)}(\zeta_n)_{l_n, l_1} \right). \quad (36)$$

The commutation relation (29) implies

$$\begin{aligned} \sum_{l'_{k+1}} W \left( \begin{matrix} l_k & l'_{k+1} \\ l_{k+1} & l_{k+2} \end{matrix} \middle| \zeta_k / \zeta_{k+1} \right) F_{l_1, \dots, l_k, l'_{k+1}, l_{k+2}, \dots, l_n}^{(i)}(\zeta_1, \dots, \zeta_k, \zeta_{k+1}, \dots, \zeta_n) \\ = F_{l_1, \dots, l_n}^{(i)}(\zeta_1, \dots, \zeta_{k+1}, \zeta_k, \dots, \zeta_n). \end{aligned} \quad (37)$$

If we set

$$F^{(i)}(\zeta_1, \dots, \zeta_n) = \sum_{l_1, \dots, l_n} F_{l_1, \dots, l_n}^{(i)}(\zeta_1, \dots, \zeta_n) [l_1, \dots, l_n], \quad (38)$$

then (37) reads as

$$W_{k+1}(\zeta_k / \zeta_{k+1}) F^{(i)}(\zeta_1, \dots, \zeta_n) = F^{(i)}(\zeta_1, \dots, \zeta_{k+1}, \zeta_k, \dots, \zeta_n). \quad (39)$$

Using the cyclic property of the trace and the commutation relations we have the difference equation:

$$\begin{aligned} F^{(i)}(\zeta_1, \dots, x^2 \zeta_k, \dots, \zeta_n) &= W_k(\zeta_{k-1} / x^2 \zeta_k)^{-1} \cdots W_2(\zeta_1 / x^2 \zeta_k)^{-1} \times \\ &\quad C W_n(\zeta_k / \zeta_n) \cdots W_{k+1}(\zeta_k / \zeta_{k+1}) F^{(i+1)}(\zeta_1, \dots, \zeta_n), \end{aligned} \quad (40)$$

where  $C$  is such that  $C[l_1, \dots, l_n] = [l_n, l_1, \dots, l_{n-1}]$ .

## Construction by representation theory

We now proceed to the mathematical construction of VO's. We shall use the following notations [16]:

$U = U_q(\widehat{sl_2})$  denotes the quantized affine algebra of type  $A_1^{(1)}$  with  $q = -x$  as the deformation parameter.

The Chevalley generators are written as  $e_i, f_i, t_i = q^{h_i}$  ( $i = 0, 1$ ) and  $q^d$ .

$U'$  will denote the subalgebra of  $U$  generated by  $e_i, f_i, t_i$  ( $i = 0, 1$ ).

$V(\lambda)$  is an irreducible  $U$ -module with highest weight  $\lambda$  and highest weight vector  $|\lambda\rangle$ .  $\Lambda_0, \Lambda_1$  denote the fundamental weights.

$V$  is a two-dimensional representation of  $U_q(sl_2)$  with a standard basis:  $\{v_+, v_-\}$ , and  $V_z$  denotes the affinization of  $V$  with the spectral parameter  $z$ . Note that the grading operator  $d$  has a well-defined action on  $V(\lambda)$  or  $V_z$ . Finally  $P\overline{R}(z)$  is the  $R$ -matrix, i.e. the intertwiner  $P\overline{R}(z_1/z_2) : V_{z_1} \otimes V_{z_2} \rightarrow V_{z_2} \otimes V_{z_1}$ , which is normalized in such a way that it sends  $v_+ \otimes v_+$  to itself.

In contrast with the situation in the case of the XXZ spin chain, and the 8-vertex models, here we are interested in VO's of the face-type. The building blocks of the latter are still the VO's of the vertex type:

$$\tilde{\Phi}_\lambda^{\mu V}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z. \quad (41)$$

Here  $\lambda, \mu$  are dominant integral weights of level  $k \geq 1$ . Strictly speaking we need to complete the spaces properly; but in the sequel we shall not dwell upon this point (see [7], [9]). A non-trivial VO of vertex type (41) exists only when  $\mu = \lambda_\pm \stackrel{def}{=} \lambda \pm (\Lambda_1 - \Lambda_0)$ , in which case it is unique up to a scalar. We normalize them as

$$\tilde{\Phi}_\lambda^{\lambda_\pm}(z)|\lambda\rangle = |\lambda_\pm\rangle + O(z)$$

where the components of (41) are defined by  $\tilde{\Phi}_\lambda^{\mu V}(z) = \sum_\varepsilon \tilde{\Phi}_\lambda^\mu(z)_\varepsilon \otimes v_\varepsilon$ . They enjoy the basic properties ([16], Appendix 2):

$$z^d \circ \tilde{\Phi}_\lambda^\mu(z')_\varepsilon \circ z^{-d} = \tilde{\Phi}_\lambda^\mu(z'/z)_\varepsilon, \quad (42)$$

$$\begin{aligned} & P\overline{R}(z_1/z_2) \tilde{\Phi}_\mu^{\nu V}(z_1) \tilde{\Phi}_\lambda^{\mu' V}(z_2) \\ &= \sum_{\mu'} \tilde{\Phi}_{\mu'}^{\nu V}(z_2) \tilde{\Phi}_\lambda^{\mu' V}(z_1) \overline{W}_k^1 \left( \begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array} \middle| z_1/z_2 \right) \frac{\psi_k(pz_2/z_1)}{\psi_k(pz_1/z_2)}, \end{aligned} \quad (43)$$

$$\sum_\varepsilon x^{(\pm 1 + \varepsilon)/2} \tilde{\Phi}_{\lambda_\pm}^\nu(x^{-2}z)_{-\varepsilon} \tilde{\Phi}_\lambda^{\lambda_\pm}(z)_\varepsilon = \delta_{\lambda\nu} g_\lambda^{\lambda_\pm} \times \text{id}_{V(\lambda)}. \quad (44)$$

Here we have set  $p = x^{2(k+2)}$ ,

$$\begin{aligned} \psi_k(z) &= \frac{(x^4 z; x^4, p)_\infty (z; x^4, p)_\infty}{(x^2 z; x^4, p)_\infty^2}, \\ g_\lambda^{\lambda_\pm} &= \frac{1}{\psi_k(p)} \frac{(p^{r_\mp} x^2; p)_\infty}{(p^{r_\mp}; p)_\infty}, \end{aligned} \quad (45)$$

and if  $\lambda = (k - a)\Lambda_0 + a\Lambda_1$  with  $\Lambda_0, \Lambda_1$  being the fundamental weights, then  $r_- = (a + 1)/(k + 2)$ ,  $r_+ = 1 - r_-$ . The formulas for  $\overline{W}_k^1$  are given in [16], eq.(A2.2) wherein  $q = -x$ .

In the particular case where  $\lambda$  has level  $k = 1$ , (41) gives rise to an isomorphism [7],[9]. In this case (43) specializes to

$$\begin{aligned} & P\overline{R}(z_1/z_2) \tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z_1) \tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1} V}(z_2) \\ &= \tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z_2) \tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1} V}(z_1) z^{1-i} \frac{(x^4 z; x^4)_\infty}{(x^4 z^{-1}; x^4)_\infty} \frac{(x^2 z^{-1}; x^4)_\infty}{(x^2 z; x^4)_\infty} \quad (z = z_1/z_2), \end{aligned}$$

and  $g_{\Lambda_i}^{\Lambda_{i+1}} = g_{\Lambda_{i+1}}^{\Lambda_i} = (x^4; x^4)_\infty / (x^2; x^4)_\infty$ . Here and till the end of this subsection,  $i$  is understood to be either 0 or 1.

*Erratum.* In [16], the  $r_\pm$  in the right hand sides of eqs.(A2.3–4) are to be corrected to  $r_\mp$ .



Fix  $L = k+2$ ,  $\xi = (L-m-2)\Lambda_0 + (m-1)\Lambda_1$ . Set  $\lambda_l = (L-l-1)\Lambda_0 + (l-1)\Lambda_1$ . Define

$$\begin{aligned}\overline{\mathcal{H}}^{(i)} &= \oplus_{l=1}^{k+1} \overline{\mathcal{H}}_l^{(i)}, \\ \overline{\mathcal{H}}_l^{(i)} &= \text{Hom}_{U'}(V(\lambda_l), V(\xi) \otimes V(\Lambda_i)).\end{aligned}\quad (46)$$

Namely,  $\overline{\mathcal{H}}_l^{(i)}$  is the space of all  $U'$ -linear maps (i.e. those commuting with the action of  $U'$ )  $f : V(\lambda_l) \rightarrow V(\xi) \otimes V(\Lambda_i)$ . Because of the  $U'$ -linearity,  $f$  is completely determined by the image, say  $v = f(u_{\lambda_l})$ , of the highest weight vector  $u_{\lambda_l}$  in  $V(\lambda_l)$ . Conversely, any vector  $v$  in  $V(\xi) \otimes V(\Lambda_i)$ , such that it is killed by the  $e_i$  and has the weight  $\lambda_l$  (modulo the null root), gives rise to such a  $U'$ -linear map. Thus we have the identification

$$\overline{\mathcal{H}}_l^{(i)} = \{v \in V(\xi) \otimes V(\Lambda_i) \mid e_j v = 0, \ t_j v = q^{\langle h_j, \lambda_l \rangle} v \ \forall j = 0, 1\}.$$

The definition of the vertex operators of face type is best described in the language of maps, rather than vectors.

The VO's of face type,

$$\tilde{\Phi}^{(i+1,i)}(z)_{l',l} : \overline{\mathcal{H}}_l^{(i)} \longrightarrow \overline{\mathcal{H}}_{l'}^{(i+1)},$$

are defined as follows. Given  $f \in \overline{\mathcal{H}}_l^{(i)}$ , consider the composition

$$\begin{aligned}V(\lambda_l) &\xrightarrow{f} V(\xi) \otimes V(\Lambda_i) \\ id \otimes \tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1} V}(z) &\xrightarrow{\quad} V(\xi) \otimes V(\Lambda_{i+1}) \otimes V_z \\ &\simeq \left( \oplus_{l'=1}^{L-1} \overline{\mathcal{H}}_{l'}^{(i+1)} \otimes V(\lambda_{l'}) \right) \otimes V_z.\end{aligned}\quad (47)$$

From (47) it follows that there exist unique maps  $f_{l'} \in \overline{\mathcal{H}}_{l'}^{(i+1)}$  such that the following is true:

$$\left( id_{V(\xi)} \otimes \tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1} V}(z) \right) \circ f = \sum_{1 \leq l' \leq L-1} (f_{l'} \otimes id_V) \circ \tilde{\Phi}_{\lambda_{l'}}^{\lambda_{l'} V}(z). \quad (48)$$

We define  $\tilde{\Phi}^{(i+1,i)}(z)_{l',l}$  by

$$\tilde{\Phi}^{(i+1,i)}(z)_{l',l}(f) = f_{l'}. \quad (49)$$

It is immediate to see that  $\tilde{\Phi}^{(i+1,i)}(z)_{l',l}$  is nonzero only when  $l' = l \pm 1$  and  $1 \leq l' \leq L-1$ .

The following relations are simple consequences of the defining relations (48), (49) and the corresponding properties (42), (43), (44) of the VO of vertex type.

$$z^d \circ \tilde{\Phi}^{(i+1,i)}(z')_{l',l} \circ z^{-d} = \tilde{\Phi}^{(i+1,i)}(z'/z)_{l',l}, \quad (50)$$

$$\begin{aligned}
& \widetilde{\Phi}^{(i,i+1)}(z_2)_{l_4,l_1} \widetilde{\Phi}^{(i+1,i)}(z_1)_{l_1,l_2} = \sum_{l_3} \widetilde{\Phi}^{(i,i+1)}(z_1)_{l_4,l_3} \widetilde{\Phi}^{(i+1,i)}(z_2)_{l_3,l_2} \\
& \quad \times \left( \frac{z_1}{z_2} \right)^{i-1} \widetilde{W} \left( \begin{matrix} l_4 & l_3 \\ l_1 & l_2 \end{matrix} \middle| z_1/z_2 \right), \tag{51} \\
& \sum_{\pm} x^{(1 \mp 1)/2} g_{\lambda_l}^{\lambda_{l \pm 1}} \widetilde{\Phi}^{(i,i+1)}(zx^{-2})_{l,l \pm 1} \widetilde{\Phi}^{(i+1,i)}(z)_{l \pm 1,l} = x^i g_{\Lambda_i}^{\Lambda_{i+1}} \times \text{id}. \tag{52}
\end{aligned}$$

Here

$$\begin{aligned}
\widetilde{W} \left( \begin{matrix} l_4 & l_3 \\ l_1 & l_2 \end{matrix} \middle| z \right) &= \frac{\overline{W}_k^1 \left( \begin{matrix} \lambda_{l_2} & \lambda_{l_3} \\ \lambda_{l_1} & \lambda_{l_4} \end{matrix} \middle| z \right)}{\overline{W}_1^1 \left( \begin{matrix} \Lambda_i & \Lambda_{i+1} \\ \Lambda_{i+1} & \Lambda_i \end{matrix} \middle| z \right)} \frac{\psi_k(pz^{-1})}{\psi_k(pz)} \frac{\psi_1(x^6 z)}{\psi_1(x^6 z^{-1})} z^{1-i} \\
&= \frac{1}{\bar{\kappa}(z^{-1/2})} \overline{W}_k^1 \left( \begin{matrix} \lambda_{l_2} & \lambda_{l_3} \\ \lambda_{l_1} & \lambda_{l_4} \end{matrix} \middle| z \right) \tag{53}
\end{aligned}$$

where  $k = L - 2$ ,  $p = x^{2L}$ , and  $\bar{\kappa}(\zeta)$  is given in (14). From the formulas (A2.2), [16] one can verify that

$$\begin{aligned}
\widetilde{W} \left( \begin{matrix} l_4 & l_3 \\ l_1 & l_2 \end{matrix} \middle| \zeta^{-2} \right) &= W \left( \begin{matrix} l_4 & l_3 \\ l_1 & l_2 \end{matrix} \middle| \zeta \right) \frac{F_{l_4 l_3} F_{l_3 l_2}}{F_{l_4 l_1} F_{l_1 l_2}} \zeta^{(l_4 + l_2 - l_3 - l_1)/2 - 1}, \\
F_{l l + 1} &= \Gamma_p\left(\frac{l}{L}\right), \quad F_{l l - 1} = \Gamma_p\left(1 - \frac{l}{L}\right). \tag{54}
\end{aligned}$$

Here we used (10)-(12).

Now, we return to the ABF models. Let  $h$  ( $h \in \mathbf{C}h_0 \oplus \mathbf{C}h_1 \oplus \mathbf{C}d$ ) act on  $\overline{\mathcal{H}}^{(i)}$  by the adjoint action,  $f \in \overline{\mathcal{H}}^{(i)} \rightarrow h \circ f - f \circ h$ . It is known that the principally specialized character of  $\overline{\mathcal{H}}_l^{(i)}$ , i.e.,  $\text{tr}_{\overline{\mathcal{H}}_l^{(i)}} x^{-2\rho}$  where  $\rho = \Lambda_0 + \Lambda_1$ , is equal to  $\text{tr}_{\mathcal{H}_l^{(i)}} x^{2D^{(i)}}$ . Therefore, we identify  $\mathcal{H}_l^{(i)}$  with  $\overline{\mathcal{H}}_l^{(i)}$  and  $D^{(i)}$  with  $-\rho$ . Furthermore, with the identification

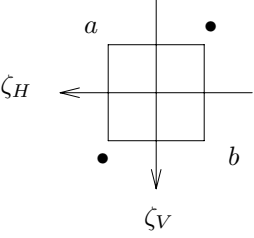
$$\Phi^{(ii')}(\zeta)_{ll'} = c^{-1} F_{ll'} \zeta^{(i' - i + l - l')/2} \widetilde{\Phi}^{(ii')}(\zeta^{-2})_{ll'}$$

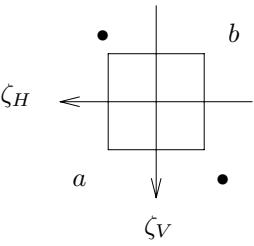
we find that the properties (28), (29), (30) are satisfied by  $\Phi^{(ii')}(\zeta)_{ll'}$  where  $c^2 = (p; x^4, p)_\infty^2 (x^4; x^4)_\infty (p; p)_\infty^2 / (px^2; x^4, p)_\infty^2 (x^2; x^4)_\infty \times (1 - p)^{(L+1)/L}$ .

## The Kashiwara-Miwa model

In this subsection, we wish to give a brief treatment of the Kashiwara-Miwa (KM) series as an example of edge-interaction models [19, 17]. The discussion of the VO's in the previous subsections can immediately be applied here, since an edge-interaction model can be regarded as a special case of a face-interaction model.

Take an integer  $N \geq 2$ . If  $N$  is even (resp. odd), set  $N = 2n$  (resp.  $2n + 1$ ). Consider the same square lattice as in the ABF model. For the KM model, we consider a local variable  $\sigma_j$  taking its values in  $\mathbf{Z}/N\mathbf{Z} \sqcup \{\bullet\}$ . For  $a \in \mathbf{Z}/N\mathbf{Z}$  we define an integer  $a^*$  such that  $0 \leq a^* \leq n$ ,  $a^* \equiv a$  or  $-a$ . The admissibility condition we impose on  $(\sigma_j, \sigma_{j'})$  is such that one and only one of  $\sigma_j, \sigma_{j'}$  takes the value  $\bullet$ . Thus a half of the variables are ‘frozen’ to the state  $\bullet$ . The Boltzmann weight is given as follows:



$$W \left( \begin{array}{c|c} a & \bullet \\ \bullet & b \end{array} \middle| \zeta \right) = \frac{\zeta^{|a^* - b^*|}}{\kappa(\zeta)} h(\zeta; a, b)$$
  


$$W \left( \begin{array}{c|c} \bullet & b \\ a & \bullet \end{array} \middle| \zeta \right) = W \left( \begin{array}{c|c} a & \bullet \\ \bullet & b \end{array} \middle| x/\zeta \right) \frac{g_b}{g_a}$$

where

$$\begin{aligned}
g_a &= \frac{\Theta_{x^{2N}}(-x^{4a^*})h(1; 0, 0)}{\Theta_{x^{2N}}(-1)h(x; 0, 0)}, \quad g_\bullet = 1, \\
h(\zeta; a, b) &= f(\zeta; a - b)f(-\zeta; a + b), \\
f(\zeta; a) &= \prod_{l=0}^{a^*-1} \Theta_{x^{2N}}(\zeta^{-1}x^{2l+1}) \prod_{l=a^*}^{n-1} \Theta_{x^{2N}}(\zeta x^{2l+1}), \\
\kappa(\zeta) &= (x^{2N}; x^{2N})_\infty^{2n} \frac{(x^2\zeta^2; x^4, x^4)_\infty (x^4\zeta^{-2}; x^4, x^4)_\infty}{(x^4\zeta^2; x^4, x^4)_\infty (x^6\zeta^{-2}; x^4, x^4)_\infty} \times \hat{\kappa}(\zeta), \\
\hat{\kappa}(\zeta) &= 1 \quad (N = 2n), \\
&= \frac{(x^{2N+2}\zeta^2; x^4, x^{4N})_\infty (x^{2N+4}\zeta^{-2}; x^4, x^{4N})_\infty}{(x^{2N}\zeta^2; x^4, x^{4N})_\infty (x^{2N+2}\zeta^{-2}; x^4, x^{4N})_\infty} \quad (N = 2n + 1).
\end{aligned} \tag{55}$$

With these Boltzmann weights, we have (16)-(18) for the KM model. (In order to adjust the crossing symmetry to the same formulation as in the ABF case, we have modified the parametrization (3.3) in [17].) The possible choices

of the boundary conditions are similar to (20), (21):

$$\begin{aligned}\sigma_{(j_1, j_2)} &= 0 && \text{if } j_1 + j_2 = i + 1 \bmod 2, \\ &= \bullet && \text{if } j_1 + j_2 = i \bmod 2.\end{aligned}$$

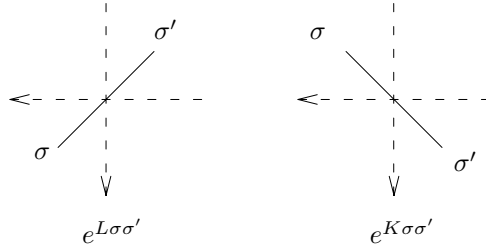
(For technical reasons we restrict to the simplest case.) Then all the consequences of these equations are equally valid in the KM model, including in particular the  $q$ -difference equations. Unlike the ABF models, however, we do not have a representation theoretic picture of the KM models. In the following section, restricting to the Ising case  $N = 2$ , we give a mathematical model of the VO's in terms of the free fermion algebra.

### 3 Vertex operators in the Ising model

The Ising model is the simplest special case of both the ABF and the KM models. In this section we wish to reexamine it in the framework of this paper. The special feature about the Ising model is that we are interested in is that one can diagonalize the CTM's using Jordan-Wigner fermions, and obtain the VO's explicitly. For the reader's convenience we shall repeat the formulation of VO's in the context of the Ising model.

#### Boltzmann weights

We choose to work in terms of the edge formulation of the model. In the literature, the Boltzmann weights are usually given in terms of the anisotropic coupling constants  $K$  and  $L$  as



We work in the ferromagnetic low-temperature regime  $K, L > 0$ . Following [3], let the coupling constants be parametrized as

$$\begin{aligned}\sinh(2K) &= -i \operatorname{sn}(iu), & \cosh(2K) &= \operatorname{cn}(iu), \\ \sinh(2L) &= i k^{-1} \operatorname{ns}(iu), & \cosh(2L) &= i k^{-1} \operatorname{ds}(iu).\end{aligned}$$

where  $u = u_V - u_H$  is the difference of (additive) spectral parameters:  $u_V$  attached to the vertical lines and  $u_H$  attached to the horizontal lines. The

$sn$  and  $cn$  are the elliptic functions with half periods  $I, iI'$  corresponding to the modulus  $k = (\sinh 2K \sinh 2L)^{-1}$  ([3], Chapter 15). We have also used Glashier's notation  $ns(u) = 1/sn(u)$ ,  $ds(u) = dn(u)/sn(u)$ , etc. The region we consider is then  $0 < k < 1$ ,  $0 < u < I'$ . Notice that changing  $u$  to  $I' - u$  has the effect of exchanging  $K$  and  $L$  (crossing symmetry).

To compare this with the KM model for  $N = 2$  (see the last figure in Section 1), let us write  $w_{\pm}(\zeta) = W \left( \begin{smallmatrix} a & \bullet \\ \bullet & b \end{smallmatrix} \middle| \zeta \right)$ ,  $\bar{w}_{\pm}(\zeta) = W \left( \begin{smallmatrix} \bullet & b \\ a & \bullet \end{smallmatrix} \middle| \zeta \right)$  ( $a, b = 0, 1$ ) where  $\pm$  is chosen according to whether  $a = b$  or  $a \neq b$ . Then we have

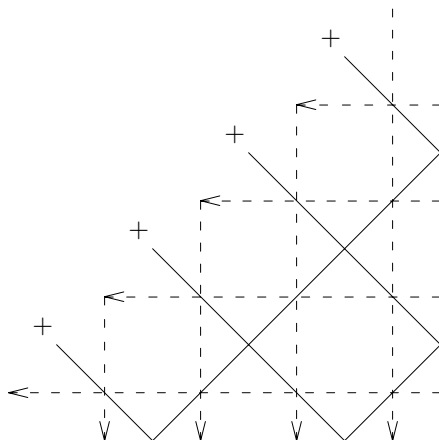
$$\begin{aligned} w_+(\zeta) &= \frac{1}{\tilde{\kappa}(\zeta)} (x^2 \zeta^2; x^8)_{\infty} (x^6 \zeta^{-2}; x^8)_{\infty}, \\ w_-(\zeta) &= \frac{1}{\tilde{\kappa}(\zeta)} \zeta (x^2 \zeta^{-2}; x^8)_{\infty} (x^6 \zeta^2; x^8)_{\infty}, \\ \bar{w}_{\pm}(\zeta) &= \frac{1}{\chi} w_{\pm}(x/\zeta), \\ \tilde{\kappa}(\zeta) &= \frac{(x^2 \zeta^2; x^4, x^4)_{\infty} (x^4 \zeta^{-2}; x^4, x^4)_{\infty}}{(x^4 \zeta^2; x^4, x^4)_{\infty} (x^6 \zeta^{-2}; x^4, x^4)_{\infty}}, \\ \chi &= \frac{(x^4; x^8)_{\infty}^2}{(x^2; x^4)_{\infty}}. \end{aligned}$$

Identifying  $x = \exp(-\pi I'/2I)$ ,  $\zeta = \exp(-\pi u/2I)$  we find  $w_-(\zeta)/w_+(\zeta) = \text{cn}(iu) + i \text{sn}(iu)$ . Hence, up to the common scalar  $\tilde{\kappa}(\zeta)$ , the KM parametrization agrees with the one for the coupling constants  $K, L$ .

## Corner transfer matrices

There are two possible ways to divide the Ising lattice into 4 quadrants. The choices depend on whether the site common to all quadrants carries a state variable or does not. Thus there are *two* CTM's to consider, and the complete space of states of the model divides into two sectors. This is analogous to the situation in the Ising CFT on the annulus, where the space of states divides into Neveu-Schwarz (NS) and Ramond (R) sectors, that are half-integrally and integrally graded, respectively. This will be made explicit below. The possibilities are:

1. The center of the lattice carries a state variable. In this case, the NW-quadrant CTM has the form



For that reason, we refer to this sector as the NS sector and write the space of states as  $\mathcal{H}^{\text{NS}} = \mathcal{H}^{\text{NS}+} \oplus \mathcal{H}^{\text{NS}-}$ , with the  $\pm$  referring to the mentioned subsectors. The corresponding CTM will be denoted by  $A_4^{\text{NS}}(\zeta) = \zeta^{D^{\text{NS}}}$ . Here  $D^{\text{NS}}$  is the CTM Hamiltonian, that the YB equations guarantee to be independent of  $\zeta$ . The subsectors  $\mathcal{H}^{\text{NS},\pm}$  are invariant under the action of  $D^{\text{NS}}$ .

2. The center of the lattice is not on the physical lattice. In this case the NW-quadrant is

Once again, by analogy with conformal field theory, we will refer to this

sector as  $\mathcal{H}^R$  and the CTM as  $A_4^R(\zeta) = \zeta^{D^R}$ .

The CTM's for the rest of the quadrants can be obtained from the above by crossing symmetry. To be precise, in the definition of the CTM's we have to make a definite choice regarding the boundary conditions at infinity. We choose the  $+$  boundary conditions.

## Diagonalization of CTM

The starting point of the diagonalization procedure is to formulate the model in terms of Jordan-Wigner (JW) fermions:

$$\begin{aligned}\psi_0^{\text{NS}} &= \sigma_1^z, & \psi_1^{\text{NS}} &= \sigma_1^y, & \psi_{2j}^{\text{NS}} &= \sigma_{j+1}^z \sigma_j^x \cdots \sigma_1^x, & \psi_{2j+1}^{\text{NS}} &= \sigma_{j+1}^y \sigma_j^x \cdots \sigma_1^x, \\ \psi_1^{\text{R}} &= \sigma_1^z, & \psi_2^{\text{R}} &= \sigma_1^y, & \psi_{2j-1}^{\text{R}} &= \sigma_j^z \sigma_{j-1}^x \cdots \sigma_1^x, & \psi_{2j}^{\text{R}} &= \sigma_j^y \sigma_{j-1}^x \cdots \sigma_1^x.\end{aligned}$$

Here the indices of the Pauli matrices refer to the lattice sites labeled by  $1, 2, \dots$  from the central site upwards or to the left. They satisfy

$$[\psi_j^{\text{NS}}, \psi_k^{\text{NS}}]_+ = 2\delta_{jk}, \quad [\psi_j^{\text{R}}, \psi_k^{\text{R}}]_+ = 2\delta_{jk}.$$

Expressed in terms of the JW fermions, the CTM Hamiltonians are

$$\begin{aligned}D^{\text{NS}} &= -\frac{iI}{\pi} \left( \sum_{j \geq 0} (2j+1) \psi_{2j+1}^{\text{NS}} \psi_{2j+2}^{\text{NS}} + k \sum_{j \geq 1} 2j \psi_{2j}^{\text{NS}} \psi_{2j+1}^{\text{NS}} \right), \\ D^{\text{R}} &= -\frac{iI}{\pi} \left( k \sum_{j \geq 0} (2j+1) \psi_{2j+1}^{\text{R}} \psi_{2j+2}^{\text{R}} + \sum_{j \geq 1} 2j \psi_{2j}^{\text{R}} \psi_{2j+1}^{\text{R}} \right)\end{aligned}$$

Since these are quadratic in the fermions, it is in principle straightforward to diagonalize them. Namely we would like to find a set of fermions

$$\phi_r^{\text{NS}} = \sum_{j=1}^{\infty} a_{rj}^{\text{NS}} \psi_j^{\text{NS}}, \quad \phi_r^{\text{R}} = \sum_{j=1}^{\infty} a_{rj}^{\text{R}} \psi_j^{\text{R}},$$

that diagonalize the adjoint action of the CTM Hamiltonians

$$[D^{\text{NS}}, \phi_r^{\text{NS}}] = -2r \phi_r^{\text{NS}}, \quad [D^{\text{R}}, \phi_r^{\text{R}}] = -2r \phi_r^{\text{R}}.$$

As it turns out the eigenvalues are labeled by  $r \in \mathbf{Z} + \frac{1}{2}$  for the  $NS$  sector and by  $r \in \mathbf{Z}$  for the  $R$  sector. We shall also require that these fermions obey the anti-commutation relations

$$[\phi_r^{\text{NS}}, \phi_s^{\text{NS}}]_+ = \eta_r \delta_{r+s,0}, \quad [\phi_r^{\text{R}}, \phi_s^{\text{R}}]_+ = \eta_r \delta_{r+s,0}, \quad \eta_r = x^{2r} + x^{-2r}, \quad (56)$$

where the normalization factor  $\eta_r$  is put in for convenience. Conversely the JW fermions can be expressed as

$$\psi_j^{\text{NS}} = \sum_r 2\eta_r^{-1} a_{-rj}^{\text{NS}} \phi_r^{\text{NS}}, \quad \psi_j^{\text{R}} = \sum_r 2\eta_r^{-1} a_{-rj}^{\text{R}} \phi_r^{\text{R}}.$$

The determination of  $a_{rj}^{\text{NS}}$  and  $a_{rj}^{\text{R}}$  is described in [2], [8]. Substituting the above expansions in the commutation relations and using the expression of the Hamiltonians in terms of JW fermions, we obtain second order linear differential equations for the generating functions

$$a_r^{\text{NS}}(t) = \sum_{j=1}^{\infty} a_{rj}^{\text{NS}} t^j, \quad a_r^{\text{R}}(t) = \sum_{j=1}^{\infty} a_{rj}^{\text{R}} t^j.$$

For (56) to make sense we demand that the coefficients  $a_{rj}^{\text{NS}}$  and  $a_{rj}^{\text{R}}$  be square summable. Changing variables by  $t = \sqrt{k} \text{sn}(v)$  and solving the equations under this regularity condition we obtain

$$a_r^{\text{NS}}(t) = \sqrt{\frac{\pi}{2I}} \left( \text{sc}(v) \cos\left(\frac{\pi r}{I} v\right) - i\sqrt{k} \text{sd}(v) \sin\left(\frac{\pi r}{I} v\right) \right), \quad (57)$$

$$a_r^{\text{R}}(t) = \sqrt{\frac{\pi}{2I}} \left( \sqrt{k} \text{sd}(v) \cos\left(\frac{\pi r}{I} v\right) - i \text{sc}(v) \sin\left(\frac{\pi r}{I} v\right) \right). \quad (58)$$

They are single-valued and holomorphic for  $|t| < 1/\sqrt{k}$ .

## The Fock spaces

The ‘Hilbert spaces’ of states  $\mathcal{H}^{\text{NS}}$  and  $\mathcal{H}^{\text{R}}$  are the Fock spaces of  $\phi_r^{\text{NS}}$  and  $\phi_r^{\text{R}}$  ( $r \neq 0$ ) over the vacuum states  $|\text{vac}\rangle_{\text{NS}}$  and  $|\text{vac}\rangle_{\text{R}}$ , respectively. The dual Fock spaces have dual vacuum states  ${}_{\text{NS}}\langle \text{vac}|$  and  ${}_{\text{R}}\langle \text{vac}|$ , normalized as usual.

The vacuum states satisfy the usual creation and annihilation conditions with respect to  $\phi_r^{\text{NS}}$  and  $\phi_r^{\text{R}}$ :

$$\begin{aligned} {}_{\text{NS}}\langle \text{vac}| \phi_r^{\text{NS}} &= 0 \quad (r < 0) & \phi_r^{\text{NS}} |\text{vac}\rangle_{\text{NS}} &= 0 \quad (r > 0), \\ {}_{\text{R}}\langle \text{vac}| \phi_r^{\text{R}} &= 0 \quad (r < 0), & \phi_r^{\text{R}} |\text{vac}\rangle_{\text{R}} &= 0 \quad (r > 0). \end{aligned} \quad (59)$$

The CTM Hamiltonians are to be normalized so that

$$D^{\text{NS}} |\text{vac}\rangle_{\text{NS}} = D^{\text{R}} |\text{vac}\rangle_{\text{R}} = {}_{\text{NS}}\langle \text{vac}| D^{\text{NS}} = {}_{\text{R}}\langle \text{vac}| D^{\text{R}} = 0.$$

They give a grading of the spaces  $\mathcal{H}^{\text{NS}}$ ,  $\mathcal{H}^{\text{R}}$  by non-negative integers:

$$\begin{aligned} \mathcal{H}^{\text{NS},+} &= \bigoplus_{d \in \mathbf{Z}_{\geq 0}, d: \text{even}} \mathcal{H}_d^{\text{NS}}, & \mathcal{H}^{\text{NS},-} &= \bigoplus_{d \in \mathbf{Z}_{\geq 0}, d: \text{odd}} \mathcal{H}_d^{\text{NS}}, \\ \mathcal{H}^{\text{R}} &= \bigoplus_{d \in \mathbf{Z}_{\geq 0}, d: \text{even}} \mathcal{H}_d^{\text{R}}, \end{aligned}$$

where  $\mathcal{H}_d^{\text{NS}} = \{v \in \mathcal{H}^{\text{NS}} \mid D^{\text{NS}} v = d v\}$  and likewise for  $\mathcal{H}_d^{\text{R}}$ .

Besides the creation/annihilation operators, the fermion algebra in each sector contains an extra one  $\psi_0^{\text{NS}}$ ,  $\phi_0^{\text{R}}$ : they commute with the CTM’s, anticommute



with the creation-annihilation fermions, and satisfy  $(\psi_0^{\text{NS}})^2 = 1$ ,  $(\phi_0^{\text{R}})^2 = 1$ . Hence their action should fix the vacuum vector up to sign. The choice of the + boundary condition corresponds to

$$\begin{aligned}\psi_0^{\text{NS}}|\text{vac}\rangle_{\text{NS}} &= |\text{vac}\rangle_{\text{NS}}, & \text{NS}\langle\text{vac}|\psi_0^{\text{NS}} &= \text{NS}\langle\text{vac}|, \\ \phi_0^{\text{R}}|\text{vac}\rangle_{\text{R}} &= |\text{vac}\rangle_{\text{R}}, & \text{R}\langle\text{vac}|\phi_0^{\text{R}} &= \text{R}\langle\text{vac}|.\end{aligned}$$

In particular  $\mathcal{H}^{\text{NS},\pm}$  is the eigenspace of  $\psi_0^{\text{NS}}$  with the eigenvalue  $\pm 1$ .

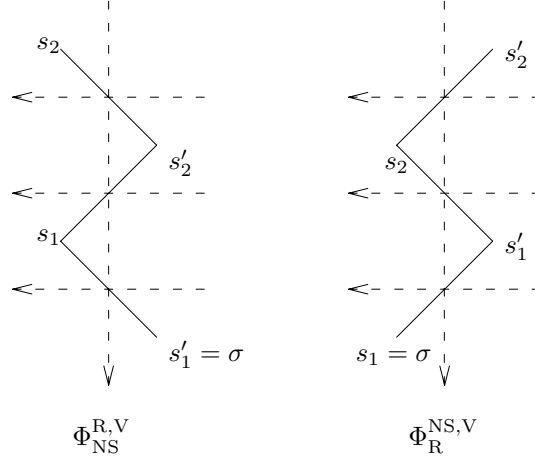
## Vertex operators

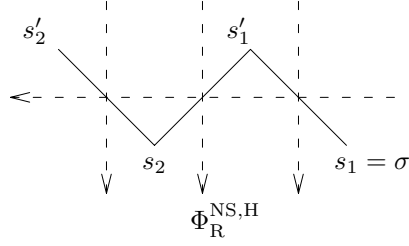
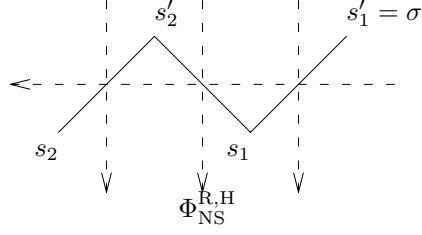
As for CTM's, there are *two* types of VO's depending on which sector they act on:

$$\begin{aligned}\Phi_{\text{NS}}^{\text{RV}}(\zeta) : \mathcal{H}^{\text{NS}} &\longrightarrow \mathcal{H}^{\text{R}}, \\ \Phi_{\text{R}}^{\text{NSV}}(\zeta) : \mathcal{H}^{\text{R}} &\longrightarrow \mathcal{H}^{\text{NS}}.\end{aligned}$$

Here the superscript V indicates that the VO is ‘vertical’. Sometimes the NS-subsectors are indicated by  $\sigma = \pm$ ; thus  $\Phi_{\text{NS}\sigma}^{\text{RV}}(\zeta) = \Phi_{\text{NS}}^{\text{RV}}(\zeta)|_{\mathcal{H}^{\text{NS}\sigma}}$ ,  $\Phi_{\text{R}}^{\text{NS},\sigma,V}(\zeta) = P^\sigma \Phi_{\text{R}}^{\text{NSV}}(\zeta)$  where  $P^\sigma$  denotes the projection onto  $\mathcal{H}^{\text{NS}\sigma}$ .

There are also the ‘horizontal’ versions of VO's  $\Phi_{\text{NS}\sigma}^{\text{RH}}(\zeta)$ ,  $\Phi_{\text{R}}^{\text{NS}\sigma\text{H}}(\zeta)$ . The following figures show the  $(\mathbf{s}, \mathbf{s}')$ -elements of these VO's where  $\mathbf{s} = (s_1, s_2, \dots)$  and so on. The NS-subsectors are indicated by  $\sigma$ .





The crossing symmetry implies

$$\Phi_{NS,\sigma}^{R,H}(\zeta) = \Phi_{NS,\sigma}^{R,V}(x/\zeta), \quad \Phi_R^{NS,\sigma,H}(\zeta) = \Phi_R^{NS,\sigma,V}(x/\zeta).$$

Arguing similarly as in the previous section (the derivation of (28)), we see that the  $\zeta$ -dependence of the VO's is given by

$$\Phi_{NS,\sigma}^{R,V}(\zeta) = \zeta^{-D^R} \Phi_{NS,\sigma}^{R,V}(1) \zeta^{D^{NS}}, \quad (60)$$

$$\Phi_R^{NS,\sigma,V}(\zeta) = \zeta^{-D^{NS}} \Phi_R^{NS,\sigma,V}(1) \zeta^{D^R}. \quad (61)$$

We will adopt the normalization

$${}_R \langle \text{vac} | \Phi_{NS,+}^{R,V}(1) | \text{vac} \rangle_{NS} = 1, \quad {}_{NS} \langle \text{vac} | \Phi_R^{NS,+,V}(1) | \text{vac} \rangle_R = 1. \quad (62)$$

## Intertwining Property

The Boltzmann weights enjoy the initial condition

$$w_{\pm}(1) = 1, \quad \bar{w}_+(1) = 1, \quad \bar{w}_-(1) = 0.$$

Hence at  $\zeta = 1$  the VO's become particularly simple: e.g.  $\Phi_R^{NS,V}(1)$  carries a vertical spin configuration to an identical one on the next left column. However it is *not* the identity because the domain  $\mathcal{H}^R$  and the range  $\mathcal{H}^{NS}$  are different

spaces. Using the JW fermions in each sector, the preservation of spins can be equivalently stated as the intertwining property

$$\psi_j^{\text{NS}} \Phi_{\text{R}}^{\text{NS},\sigma,\text{V}}(1) = \Phi_{\text{R}}^{\text{NS},-\sigma,\text{V}}(1) \psi_{j+1}^{\text{R}} \quad (j \geq 1), \quad (63)$$

$$\sigma \Phi_{\text{R}}^{\text{NS},\sigma,\text{V}}(1) = \Phi_{\text{R}}^{\text{NS},\sigma,\text{V}}(1) \psi_1^{\text{R}}. \quad (64)$$

In the same manner we find the following for the other types of VO's:

$$\psi_j^{\text{R}} \Phi_{\text{NS},\sigma}^{\text{R},\text{V}}(1) = \Phi_{\text{NS},-\sigma}^{\text{R},\text{V}}(1) \psi_{j+1}^{\text{NS}} \quad (j \geq 1),$$

$$i\sigma \Phi_{\text{NS},\sigma}^{\text{R},\text{V}}(1) = \Phi_{\text{NS},-\sigma}^{\text{R},\text{V}}(1) \psi_1^{\text{NS}},$$

$$\psi_{j+1}^{\text{NS}} \Phi_{\text{R}}^{\text{NS},\sigma,\text{H}}(1) = \Phi_{\text{R}}^{\text{NS},-\sigma,\text{H}}(1) \psi_j^{\text{R}} \quad (j \geq 1),$$

$$\psi_1^{\text{NS}} \Phi_{\text{R}}^{\text{NS},-\sigma,\text{H}}(1) = -i\sigma \Phi_{\text{R}}^{\text{NS},\sigma,\text{H}}(1),$$

$$\psi_{j+1}^{\text{R}} \Phi_{\text{NS},\sigma}^{\text{R},\text{H}}(1) = \Phi_{\text{NS},-\sigma}^{\text{R},\text{H}}(1) \psi_j^{\text{NS}} \quad (j \geq 1),$$

$$\psi_1^{\text{R}} \Phi_{\text{NS},\sigma}^{\text{R},\text{H}}(1) = \sigma \Phi_{\text{NS},\sigma}^{\text{R},\text{H}}(1).$$

Along with the normalization condition (62), these commutation relations with the fermions characterize the VO's uniquely, and all their matrix elements can be determined. The working is described in Appendix A. We obtain the following formulas for the general matrix elements:

$$\begin{aligned} & {}_{\text{R}}\langle \text{vac} | \phi_{l_1}^{\text{R}} \cdots \phi_{l_n}^{\text{R}} \Phi_{\text{NS}}^{\text{R},\text{V}}(\zeta) \phi_{-k_1}^{\text{NS}} \cdots \phi_{-k_m}^{\text{NS}} | \text{vac} \rangle_{\text{NS}} / \zeta^{2(-l_1 - \dots - l_n + k_1 + \dots + k_m)} \\ &= \prod x^{2l_j} \gamma_{l_j} \prod \left( \sqrt{-1} x^{-1/2} \gamma_{k_i-1/2} \right) \times \prod_{j < j'} X_{l_j, l_{j'}} \prod_{i, j} X_{l_j, -k_i} \prod_{i < i'} X_{k_i, k_{i'}}, \end{aligned} \quad (65)$$

$$\begin{aligned} & {}_{\text{NS}}\langle \text{vac} | \phi_{k_m}^{\text{NS}} \cdots \phi_{k_1}^{\text{NS}} \Phi_{\text{R}}^{\text{NS},\text{V}}(\zeta) \phi_{-l_n}^{\text{R}} \cdots \phi_{-l_1}^{\text{R}} | \text{vac} \rangle_{\text{NS}} / \zeta^{2(l_1 + \dots + l_n - k_1 - \dots - k_m)} \\ &= \prod \gamma_{l_j} \prod \left( -\sqrt{-1} x^{2k_i-1/2} \gamma_{k_i-1/2} \right) \times \prod_{j < j'} X_{l_j, l_{j'}} \prod_{i, j} X_{l_j, -k_i} \prod_{i < i'} X_{k_i, k_{i'}}. \end{aligned} \quad (66)$$

Here we have set

$$X_{k,l} = \frac{x^{4k} - x^{4l}}{1 - x^{4(k+l)}} = -X_{l,k}, \quad X_{-k,-l} = X_{k,l}, \quad \gamma_n = \frac{(x^2; x^4)_n}{(x^4; x^4)_n}. \quad (67)$$

## Vacuum-to-vacuum two point functions

In what follows, we shall use the shorthand notations

$$\Phi_{\text{R}}^{\text{NS},\sigma,V}(\zeta) = \Phi^{\sigma}(\zeta), \quad \Phi_{\text{NS},\sigma}^{\text{R},V}(\zeta) = \Phi_{\sigma}(\zeta).$$

When no confusion may arise, we shall also drop  $\text{NS}$ ,  $\text{R}$  and write  $|\text{vac}\rangle$  for either  $|\text{vac}\rangle_{\text{NS}}$  or  $|\text{vac}\rangle_{\text{R}}$ , etc.

Using the formulas for the matrix elements (65)-(66) one can in principle compute the vacuum expectation values of products of VO's. The expansion in powers of  $x$  suggests that the two point functions are given by the following infinite products: setting  $\zeta = \zeta_1/\zeta_2$  we have

$$\sum_{\sigma} \langle \text{vac} | \Phi_{\sigma}(\zeta_1) \Phi^{\sigma}(\zeta_2) | \text{vac} \rangle = \frac{(x^3 \zeta; x^4)_{\infty}}{(x \zeta; x^4)_{\infty}} F_{+}(\zeta^2), \quad (68)$$

$$\langle \text{vac} | \Phi^{+}(\zeta_1) \Phi_{+}(\zeta_2) | \text{vac} \rangle = F_{-}(\zeta^2), \quad (69)$$

$$\langle \text{vac} | \phi_{1/2}^{\text{NS}} \Phi^{-}(\zeta_1) \Phi_{-}(\zeta_2) \phi_{-1/2}^{\text{NS}} | \text{vac} \rangle = (\zeta^{-1} + \zeta) F_{-}(\zeta^2), \quad (70)$$

$$\langle \text{vac} | \Phi^{+}(\zeta_1) \Phi_{-}(\zeta_2) \phi_{-1/2}^{\text{NS}} | \text{vac} \rangle = i x^{-1/2} \zeta_2 F_{+}(\zeta^2), \quad (71)$$

$$\langle \text{vac} | \phi_{1/2}^{\text{NS}} \Phi^{-}(\zeta_1) \Phi_{+}(\zeta_2) | \text{vac} \rangle = -i x^{1/2} \zeta_1^{-1} F_{+}(\zeta^2). \quad (72)$$

Here

$$F_{+}(z) = \frac{(x^2 z; x^4, x^8)_{\infty}^2}{(x^2 z; x^4)_{\infty} (x^4 z; x^4, x^4)_{\infty}},$$

$$F_{-}(z) = \frac{(x^6 z; x^4, x^8)_{\infty}^2}{(x^4 z; x^4, x^4)_{\infty}}.$$

We shall derive these equations using the formulation of the Ising model as a special case of the ABF models (see the end of this section).

*Remark.* In the language of the string theory, the vertex operator  $\Phi_{\text{NS}}^{\text{R},V}(z^{-1/2})$ , for example, is expressed as

$$\begin{aligned} \Phi_{\text{NS}}^{\text{R}}(z^{-1/2}) &= {}_{\text{R}}\langle \text{vac} | : \exp Y : | \text{vac} \rangle_{\text{NS}}, \\ Y &= -\frac{1}{2} \sum_{m,n>0} X_{m,n} \varphi_{-m}^{\text{R}} \varphi_{-n}^{\text{R}} z^{m+n} + \sum_{m,r>0} X_{m,-r} \varphi_{-m}^{\text{R}} \varphi_r^{\text{NS}} z^{m-r} \\ &\quad - \phi_0^{\text{R}} \left( \sum_{m>0} \varphi_{-m}^{\text{R}} z^m - \sum_{r>0} \varphi_r^{\text{NS}} z^{-r} \right) - \frac{1}{2} \sum_{r,s>0} X_{r,s} \varphi_r^{\text{NS}} \varphi_s^{\text{NS}} z^{-r-s}, \end{aligned}$$

where we have set, for  $m, n \in \mathbf{Z}$  and  $r, s \in \mathbf{Z} + \frac{1}{2}$ ,

$$\varphi_{-m}^{\text{R}} = x^{2m} \frac{\gamma_m}{\eta_m} \phi_{-m}^{\text{R}}, \quad \varphi_r^{\text{NS}} = \sqrt{-1} x^{-1/2} \frac{\gamma_{r-\frac{1}{2}}}{\eta_r} \phi_r^{\text{NS}}.$$

For the notation  ${}_{\text{R}}\langle \text{vac} | : \exp Y : | \text{vac} \rangle_{\text{NS}}$ , see the literature on the fermion emission vertex, for example [20]. For  $x = 1$ , the vertex operators  $\Phi^{\sigma}(z^{-1/2})$

and  $\Phi_\sigma(z^{-1/2})$  are the primary fields with the conformal weight  $1/16$  of the  $c = 1/2$  minimal model, up to some power of  $z$ . It can be verified, by noting the identity below, that (68) and (69) correspond to the four and two point functions of spin fields respectively [6]:

$$\begin{aligned} & \frac{(x^3\zeta; x^4)_\infty (x^2\zeta^2; x^8)_\infty}{(x\zeta; x^4)_\infty (x^6\zeta^2; x^8)_\infty} \\ &= \phi \left( \begin{matrix} 1/4 & -1/4 \\ 1/2 \end{matrix}; x^8, x^6\zeta^2 \right) + \frac{x\zeta}{1+x^2} \phi \left( \begin{matrix} 1/4 & 3/4 \\ 3/2 \end{matrix}; x^8, x^6\zeta^2 \right), \end{aligned}$$

where

$$\phi \left( \begin{matrix} a & b \\ c \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(q^a; q)_n (q^b; q)_n}{(q^c; q)_n (q; q)_n} z^n, \quad (z; q)_n = \prod_{j=0}^{n-1} (1 - zq^j),$$

denotes the basic hypergeometric series.

## Unitarity and commutation relations

We have the following relations:

$$\sum_{\sigma} \Phi_{\sigma}(x\zeta) \Phi^{\sigma}(\zeta) = g^R \times \text{id}_{\mathcal{H}^R}, \quad (73)$$

$$\Phi^{\sigma}(x\zeta) \Phi_{\sigma}(\zeta) = g^{\text{NS}} \times \text{id}_{\mathcal{H}^{\text{NS}, \sigma}}. \quad (74)$$

The reason is that, from the intertwining properties of the VO's, the LHS can be shown to commute with the fermions. Since our Fock spaces are irreducible, they must then be acting as scalars. The scalars can be determined by considering the vacuum-to-vacuum expectation values given above. In this way we get

$$\begin{aligned} g^R &= \frac{(x^4; x^4, x^8)_{\infty}^2}{(x^2; x^4, x^4)_{\infty}}, & g^{\text{NS}} &= \frac{(x^8; x^4, x^8)_{\infty}^2}{(x^6; x^4, x^4)_{\infty}}, \\ \frac{g^R}{g^{\text{NS}}} &= \frac{\text{tr}_{\mathcal{H}^{\text{NS}}} (x^{2D})}{\text{tr}_{\mathcal{H}^R} (x^{2D})} = \frac{(-x^2; x^4)_{\infty}}{(-x^4; x^4)_{\infty}}. \end{aligned}$$

Using a similar reasoning we can derive from (68)–(72) the following commutation relations:

$$\Phi^{\sigma}(\zeta_2) \Phi_{\sigma'}(\zeta_1) = \Phi^{\sigma}(\zeta_1) \Phi_{\sigma'}(\zeta_2) w_{\sigma\sigma'}(\zeta_1/\zeta_2), \quad (75)$$

$$\Phi_{\sigma}(\zeta_2) \Phi^{\sigma}(\zeta_1) = \sum_{\sigma'} \Phi_{\sigma'}(\zeta_1) \Phi^{\sigma'}(\zeta_2) \bar{w}_{\sigma\sigma'}(\zeta_1/\zeta_2). \quad (76)$$

which are what we expect.

## Difference equation

Put

$$\begin{aligned} G_{\pm}^R(\zeta_1/\zeta_2) &= \text{tr}_{\mathcal{H}^R} \left( x^{2D} (\Phi_+(\zeta_1)\Phi^+(\zeta_2) \pm \Phi_-(\zeta_1)\Phi^-(\zeta_2)) \right) = G_+^R(\pm\zeta_1/\zeta_2), \\ G_{\pm}^{\text{NS}}(\zeta_1/\zeta_2) &= \text{tr}_{\mathcal{H}^{\text{NS}}} \left( x^{2D} (\Phi^+(\zeta_1)\Phi_+(\zeta_2) \pm \Phi^-(\zeta_1)\Phi_-(\zeta_2)) \right) = G_+^{\text{NS}}(\pm\zeta_1/\zeta_2). \end{aligned}$$

From the commutation relations we then deduce that

$$\begin{aligned} w_+(\zeta)G_{\pm}^{\text{NS}}(\zeta) &= G_{\pm}^R(x^2\zeta), \\ G_{\pm}^R(x^4\zeta) &= w_+(x^2\zeta) (\bar{w}_+(\zeta) \pm \bar{w}_-(\zeta)) G_{\pm}^R(\zeta). \end{aligned}$$

The coefficient in the second equation can be factorized:

$$\begin{aligned} \bar{w}_+(\zeta) \pm \bar{w}_-(\zeta) &= \frac{(\pm x\zeta; x^4)_{\infty}}{(\pm x^3\zeta; x^4)_{\infty}} \frac{(\pm x^3\zeta^{-1}; x^4)_{\infty}}{(\pm x\zeta^{-1}; x^4)_{\infty}} \\ &\quad \times \frac{(x^2\zeta^2; x^4)_{\infty}}{(x^2\zeta^{-2}; x^4)_{\infty}} \frac{(x^4\zeta^2; x^4, x^4)_{\infty}}{(x^4\zeta^{-2}; x^4, x^4)_{\infty}} \frac{(x^2\zeta^{-2}; x^4, x^8)_{\infty}^2}{(x^2\zeta^2; x^4, x^8)_{\infty}^2}. \end{aligned}$$

Using this we find the following solution:

$$\begin{aligned} G_{\pm}^R(\zeta) &= C \frac{(\pm x^3\zeta; x^4, x^4)_{\infty}}{(\pm x\zeta; x^4, x^4)_{\infty}} \frac{(\pm x^7\zeta^{-1}; x^4, x^4)_{\infty}}{(\pm x^5\zeta^{-1}; x^4, x^4)_{\infty}} \frac{f(\zeta)}{f(x)}, \\ f(\zeta) &= \frac{(x^2\zeta^2; x^4, x^8, x^8)_{\infty}^4}{(x^4\zeta^2; x^4, x^4, x^8)_{\infty}^2} \frac{(x^{10}\zeta^{-2}; x^4, x^8, x^8)_{\infty}^4}{(x^{12}\zeta^{-2}; x^4, x^4, x^8)_{\infty}^2} \frac{(x^4\zeta^2; x^4, x^8)_{\infty}}{(x^2\zeta^2; x^4, x^8)_{\infty}^3 (x^8\zeta^{-2}; x^2, x^8)_{\infty}}. \end{aligned}$$

The normalization constant  $C$  is determined by setting  $\zeta = x$  and using the properties (73).

$$C = (x^2; x^4)_{\infty} g^R \text{tr}_{\mathcal{H}^R} (x^{2D}) = \frac{(x^4; x^4, x^4)_{\infty}}{(x^6; x^4, x^4)_{\infty}}.$$

Since the solution of a difference equation is determined only up to pseudo-constants, it is necessary to verify (e.g. by examining the analyticity) that the above solution does give the quantity defined through the trace. We have not proved this but checked it to some order in  $x$ .

## 8 vertex model at the Ising point

As is well known, the 8 vertex model at a special value of the parameters can be regarded as the superposition of two non-interacting Ising models. Let us consider a checkerboard lattice whose faces are shaded alternatingly. Place two kinds of Ising spins on the faces, one on the shaded faces and the other on the unshaded ones. In both cases we assume the  $+$  boundary condition. Place also 8 vertex ‘spin’ variables on the edges taking values  $\pm$ . Consider a vertical (resp.

horizontal) edge separating two faces that carry the spin variables  $\sigma_1, \sigma_2$ . We let the edge variable  $\varepsilon$  have the value  $\sigma_1\sigma_2$  if the left (resp. upper) face is shaded, and  $-\sigma_1\sigma_2$  if otherwise. In this way the Ising configurations are mapped to the 8 vertex configurations with anti-ferroelectric boundary condition.

The space of states for this doubled model is the tensor product of the Fock spaces  $\mathcal{H}^0 = \mathcal{H}^{\text{NS}} \otimes \mathcal{H}^{\text{R}}$ ,  $\mathcal{H}^1 = \mathcal{H}^{\text{R}} \otimes \mathcal{H}^{\text{NS}}$ . This can be seen by combining two Ising configurations to form an 8-vertex configuration: the Ising configurations must belong to two different sectors.

The following operators act on the doubled space  $\mathcal{H}^0$ :  $\phi_r^{\text{N}} \otimes \text{id}, \psi_0^{\text{NS}} \otimes \phi_r^{\text{R}}$ , which are mutually anti-commuting. For brevity, we refer to them by  $\phi_r^{\text{N}}, \phi_r^{\text{R}}$ , respectively. Similar conventions apply to  $\mathcal{H}^1$ .

For  $i = 0, 1 \in \mathbf{Z}/2\mathbf{Z}$ , we define the VO's  $\Phi_{i\varepsilon}^{i+1} : \mathcal{H}^i \rightarrow \mathcal{H}^{i+1}$  by

$$\begin{aligned}\Phi_{0\varepsilon}^1(\zeta) &= \sum_{\sigma} \Phi_{\text{NS},\sigma}^{\text{RV}}(\zeta) \otimes \Phi_{\text{R}}^{\text{NS},-\varepsilon\sigma,V}(\zeta), \\ \Phi_{1\varepsilon}^0(\zeta) &= \sum_{\sigma} \Phi_{\text{R}}^{\text{NS},\sigma,V}(\zeta) \otimes \Phi_{\text{NS},\varepsilon\sigma}^{\text{R},V}(\zeta).\end{aligned}\tag{77}$$

Using the commutation relations for the Ising VO's we find that (77) satisfies

$$\Phi_{i+1\varepsilon_2}^i(\zeta_2)\Phi_{i\varepsilon_1}^{i+1}(\zeta_1) = \sum_{\varepsilon'_1, \varepsilon'_2} R(\zeta_1/\zeta_2)_{\varepsilon_1\varepsilon_2; \varepsilon'_1\varepsilon'_2} \Phi_{i+1\varepsilon'_1}^i(\zeta_1)\Phi_{i\varepsilon'_2}^{i+1}(\zeta_2).$$

The  $R$  matrix is the one given in [15] specialized to  $q = x^4$ , except that our  $\zeta^{-1}$  is the  $\zeta = \zeta_{8v}$  there; explicitly, in the notation of [15],

$$\begin{aligned}\frac{a(\zeta_{8v})}{\kappa(\zeta_{8v})} &= w_-(\zeta)\bar{w}_+(\zeta), & \frac{b(\zeta_{8v})}{\kappa(\zeta_{8v})} &= w_+(\zeta)\bar{w}_-(\zeta), \\ \frac{c(\zeta_{8v})}{\kappa(\zeta_{8v})} &= w_+(\zeta)\bar{w}_+(\zeta), & \frac{d(\zeta_{8v})}{\kappa(\zeta_{8v})} &= w_-(\zeta)\bar{w}_-(\zeta).\end{aligned}$$

The matrix elements of VO's can also be obtained in a straightforward way. We give below the generating functions of the one- and two-particle matrix elements using

$$\phi(z) = \phi^{\text{NS}}(z) + \phi^{\text{R}}(z) = \sum_{r \in \frac{1}{2}\mathbf{Z}} \phi_r z^{-r}.$$

Set  $\zeta = 1$ , and write  $\Phi_{\varepsilon} = \Phi_{0-\varepsilon}^1(1)$  or  $\Phi_{1\varepsilon}^0(1)$ .

$$\langle \Phi_+ \phi(z) \phi(w) \rangle = \frac{(x^2 z)_{\infty}}{(z)_{\infty}} \frac{(x^2 w)_{\infty}}{(w)_{\infty}} \frac{(z-w)(1-x^{-1}\sqrt{zw})}{(\sqrt{z}-x\sqrt{w})(\sqrt{z}-x^{-1}\sqrt{w})},\tag{78}$$

$$\langle \Phi_- \phi(z) \phi(w) \rangle = ix^{-1/2}(\sqrt{z}-\sqrt{w}) \frac{(x^2 z)_{\infty}}{(z)_{\infty}} \frac{(x^2 w)_{\infty}}{(w)_{\infty}},\tag{79}$$

$$\langle \phi(z) \Phi_+ \phi(w) \rangle = (1 + \frac{\sqrt{w}}{\sqrt{z}}) \frac{(x^4 z^{-1})_{\infty}}{(x^2 z^{-1})_{\infty}} \frac{(x^2 w)_{\infty}}{(w)_{\infty}},\tag{80}$$

$$\langle \phi(z)\Phi_-\phi(w) \rangle = -ix^{-1/2} \frac{1}{\sqrt{z}} \frac{(x^4 z^{-1})_\infty}{(x^2 z^{-1})_\infty} \frac{(x^2 w)_\infty}{(w)_\infty} \frac{(z-w)(x-\sqrt{zw})}{(\sqrt{z}-x\sqrt{w})(\sqrt{z}-x^{-1}\sqrt{w})}, \quad (81)$$

$$\langle \phi(z)\phi(w)\Phi_+ \rangle = -\frac{1}{\sqrt{zw}} \frac{(x^4 z^{-1})_\infty}{(x^2 z^{-1})_\infty} \frac{(x^4 w^{-1})_\infty}{(x^2 w^{-1})_\infty} \frac{(z-w)(x-\sqrt{zw})}{(\sqrt{z}-x\sqrt{w})(\sqrt{z}-x^{-1}\sqrt{w})}, \quad (82)$$

$$\langle \phi(z)\phi(w)\Phi_- \rangle = -ix^{1/2} \frac{\sqrt{z}-\sqrt{w}}{\sqrt{zw}} \frac{(x^4 z^{-1})_\infty}{(x^2 z^{-1})_\infty} \frac{(x^4 w^{-1})_\infty}{(x^2 w^{-1})_\infty}. \quad (83)$$

Here we used  $(z)_\infty = (z; x^4)_\infty$ .

## Ising model in the RSOS formulation

In this subsection we examine the Ising model formulated as an ABF model with  $L = 4$ . Using the representation theoretical construction we shall calculate the two point functions for the VOs.

For our purpose it is useful to study first the symmetry under the Dynkin diagram automorphism. Let  $\nu$  denote the algebra automorphism of  $U'$  given by  $\nu(e_i) = e_{1-i}$ ,  $\nu(f_i) = f_{1-i}$ ,  $\nu(t_i) = t_{1-i}$ . For  $\lambda = (k-l)\Lambda_0 + l\Lambda_1$  set  $\bar{\lambda} = l\Lambda_0 + (k-l)\Lambda_1$ . Then there exists a unique isomorphism (which we denote also by  $\nu$ ) of vector spaces  $V(\lambda) \rightarrow V(\bar{\lambda})$  such that  $\nu(|\lambda\rangle) = |\bar{\lambda}\rangle$  and  $\nu(x|u\rangle) = \nu(x)\nu(|u\rangle)$  for all  $x \in U'$  and  $|u\rangle \in V(\lambda)$ . To make manifest the symmetry under  $\nu$  let us introduce the VO of vertex type in the principal picture:

$$\widehat{\Phi}_\lambda^{\mu V}(\zeta) = \zeta^{(\mu-\lambda, \rho)} \times (\text{id} \otimes \zeta^{\bar{\rho}}) \widetilde{\Phi}_\lambda^{\mu V}(\zeta^2)$$

where  $\widetilde{\Phi}_\lambda^{\mu V}(z)$  is the VO (41). In our case  $\bar{\rho} = \Lambda_1 - \Lambda_0$ . The power of  $\zeta$  is supplied in order to retain the normalization  $\widehat{\Phi}_{\lambda_l}^{\lambda_l \pm 1 V}(\zeta)|\lambda_l\rangle = |\lambda_{l \pm 1}\rangle \otimes v_\mp + O(\zeta)$ . We have

$$\widehat{\Phi}_\lambda^{\bar{\mu} V}(\zeta) = (\nu \otimes \sigma^x) \circ \widehat{\Phi}_\lambda^{\mu V}(\zeta) \circ \nu^{-1}.$$

Given  $f \in \text{Hom}_{U'}(V(\lambda), V(\xi) \otimes V(\eta))$  we write  $\nu(f) = (\nu \otimes \nu) \circ f \circ \nu^{-1} \in \text{Hom}_{U'}(V(\bar{\lambda}), V(\bar{\xi}) \otimes V(\bar{\eta}))$ . Define the VO of face type in the principal picture  $\widehat{\Phi}_{\lambda'\bar{\lambda}}^{(\xi; \eta' \eta)}(\zeta)$  by

$$(\text{id}_{V(\xi)} \otimes \widehat{\Phi}_\eta^{\eta' V}(\zeta)) \circ f = \sum_{\lambda'} \left( \widehat{\Phi}_{\lambda'\bar{\lambda}}^{(\xi; \eta' \eta)}(\zeta)(f) \otimes \text{id} \right) \circ \widehat{\Phi}_\lambda^{\lambda' V}(\zeta). \quad (84)$$

Then we have the symmetry

$$\widehat{\Phi}_{\bar{\lambda}'\bar{\lambda}}^{(\bar{\xi}; \bar{\eta}' \bar{\eta})}(\zeta)(f) = \nu \left( \widehat{\Phi}_{\lambda'\bar{\lambda}}^{(\xi; \eta' \eta)}(\zeta)(\nu^{-1}(f)) \right).$$

Now we focus attention to the Ising model. Taking  $k = 2$ , we write  $\widehat{\Phi}_{ll'}^{(j; ii')}(z)$  for  $\widehat{\Phi}_{\lambda_l \lambda_{l'}}^{(\Lambda_j; \Lambda_i \Lambda_{i'})}(z)$  (recall that  $\lambda_l = (2-l)\Lambda_0 + l\Lambda_1$ ). Exhibiting the  $j$  dependence



explicitly we set

$$\mathcal{H}_l^{(j;i)} = \{v \in V(\Lambda_j) \otimes V(\Lambda_i) \mid e_s v = 0, \ t_s v = q^{\langle h_s, \lambda_l \rangle} v \quad (s = 0, 1)\},$$

so that

$$V(\Lambda_j) \otimes V(\Lambda_i) = \bigoplus_{l \equiv i+j \bmod 2} \mathcal{H}_l^{(j;i)} \otimes V(\lambda_l).$$

Clearly  $\nu$  induces the isomorphism  $\mathcal{H}_l^{(j;i)} \simeq \mathcal{H}_{2-l}^{(1-j;1-i)}$ . We shall identify

$$\mathcal{H}_0^{(0;0)} = \mathcal{H}^{\text{NS}+}, \quad \mathcal{H}_2^{(0;0)} = \mathcal{H}^{\text{NS}-}, \quad \mathcal{H}_1^{(0;1)} \simeq \mathcal{H}_1^{(1;0)} = \mathcal{H}^{\text{R}}$$

and set

$$\begin{aligned} |\text{vac}\rangle_{\text{NS}} &= |\Lambda_0\rangle \otimes |\Lambda_0\rangle \in \mathcal{H}_0^{(0;0)}, \\ |\text{vac}\rangle_{\text{R},i} &= |\Lambda_i\rangle \otimes |\Lambda_{1-i}\rangle \in \mathcal{H}_1^{(i;1-i)}. \end{aligned}$$

Defining the dual vectors in a similar manner, we wish to compute the two point functions of  $\widehat{\Phi}_{ll'}^{(j;ii')}(\zeta)$ . From their defining property (84) it follows that

$$\begin{aligned} &\langle \Lambda_0 | \widehat{\Phi}_{\Lambda_1}^{\Lambda_0 V}(\zeta_1) \widehat{\Phi}_{\Lambda_0}^{\Lambda_1 V}(\zeta_2) | \Lambda_0 \rangle \\ &= {}_{\text{NS}} \langle \text{vac} | \widehat{\Phi}_{01}^{(0;01)}(\zeta_1) \widehat{\Phi}_{10}^{(0;10)}(\zeta_2) | \text{vac} \rangle_{\text{NS}} \times \langle \lambda_0 | \widehat{\Phi}_{\lambda_1}^{\lambda_0 V}(\zeta_1) \widehat{\Phi}_{\lambda_0}^{\lambda_1 V}(\zeta_2) | \lambda_0 \rangle \end{aligned} \quad (85)$$

$$\begin{aligned} &\langle \Lambda_i | \widehat{\Phi}_{\Lambda_{1-i}}^{\Lambda_i V}(\zeta_1) \widehat{\Phi}_{\Lambda_i}^{\Lambda_{1-i} V}(\zeta_2) | \Lambda_i \rangle = \sum_{l=0,2} {}_{\text{R},1-i} \langle \text{vac} | \widehat{\Phi}_{1l}^{(1-i;i,1-i)}(\zeta_1) \widehat{\Phi}_{l1}^{(1-i;1-ii)}(\zeta_2) | \text{vac} \rangle_{\text{R},1-i} \\ &\times \langle \lambda_1 | \widehat{\Phi}_{\lambda_l}^{\lambda_1 V}(\zeta_1) \widehat{\Phi}_{\lambda_1}^{\lambda_l V}(\zeta_2) | \lambda_1 \rangle. \end{aligned} \quad (86)$$

On the other hand, the following can be obtained by solving the  $q\text{KZ}$  equation for VOs of vertex type.

$$\begin{aligned} &\langle \Lambda_i | \widehat{\Phi}_{\Lambda_{1-i}}^{\Lambda_i V}(\zeta_1) \widehat{\Phi}_{\Lambda_i}^{\Lambda_{1-i} V}(\zeta_2) | \Lambda_i \rangle = \frac{(x^6 \zeta^2; x^4)_\infty}{(x^4 \zeta^2; x^4)_\infty} (v_\pm \otimes v_\mp + x \zeta v_\mp \otimes v_\pm) \\ &\langle \lambda_l | \widehat{\Phi}_{\lambda_1}^{\lambda_l V}(\zeta_1) \widehat{\Phi}_{\lambda_l}^{\lambda_1 V}(\zeta_2) | \lambda_l \rangle = \frac{(x^8 \zeta^2; x^4, x^4)_\infty}{(x^6 \zeta^2; x^4, x^8)_\infty (x^{14} \zeta^2; x^4, x^8)_\infty} (v_\pm \otimes v_\mp + x \zeta v_\mp \otimes v_\pm) \\ &\langle \lambda_1 | \widehat{\Phi}_{\lambda_l}^{\lambda_1 V}(\zeta_1) \widehat{\Phi}_{\lambda_1}^{\lambda_l V}(\zeta_2) | \lambda_1 \rangle = \frac{(x^8 \zeta^2; x^4, x^4)_\infty}{(x^6 \zeta^2; x^4, x^8)_\infty (x^{14} \zeta^2; x^4, x^8)_\infty} \times \\ &\left( \phi \left( \begin{matrix} 1/4 & -1/4 \\ & 1/2 \end{matrix}; x^8, x^{10} \zeta^2 \right) v_\mp \otimes v_\pm + \frac{x^3 \zeta}{1+x^2} \phi \left( \begin{matrix} 1/4 & 3/4 \\ & 3/2 \end{matrix}; x^8, x^{10} \zeta^2 \right) v_\pm \otimes v_\mp \right). \end{aligned}$$

Here  $\zeta = \zeta_2/\zeta_1$ , and the sign  $\pm$  is chosen according as  $i = 0, 1$  or  $l = 0, 2$ , respectively. Using the symmetry

$${}_{\text{R},1} \langle \text{vac} | \widehat{\Phi}_{1l}^{(1;01)}(\zeta_1) \widehat{\Phi}_{l1}^{(1;10)}(\zeta_2) | \text{vac} \rangle_{\text{R},1} = {}_{\text{R},0} \langle \text{vac} | \widehat{\Phi}_{12-l}^{(0;10)}(\zeta_1) \widehat{\Phi}_{2-l1}^{(0;01)}(\zeta_2) | \text{vac} \rangle_{\text{R},0},$$

we can solve (85), (86) for the two point functions of the  $\widehat{\Phi}_{ll'}^{(j;ii')}(\zeta)$ . With the identificaiton

$$\begin{aligned}\widehat{\Phi}_{01}^{(0;01)}(\zeta) &= \Phi^+(\zeta^{-1}), & \widehat{\Phi}_{21}^{(0;01)}(\zeta) &= \Phi^-(\zeta^{-1}), \\ \widehat{\Phi}_{10}^{(0;10)}(\zeta) &= \Phi_+(\zeta^{-1}), & \widehat{\Phi}_{12}^{(0;10)}(\zeta) &= \Phi_-(\zeta^{-1}),\end{aligned}$$

we then recover the formulas given in (68), (69).

## Creation and annihilation operators

To conclude this section we shall briefly discuss the creation and annihilation operators for the row-to-row Hamiltonian. We shall introduce two kinds of such operators, and discuss the relation between them.

So far we have chosen to work with the  $+$  boundary condition for the CTM. To take the excited states into account, it is necessary to consider both of the two boundary conditions  $\sigma_j = \pm$  ( $j \gg 1$ ). We denote the space for the CTM with those boundary conditions by  $\mathcal{H}_\pm$  respectively, where  $\mathcal{H}$  refers to either  $\mathcal{H}^R$  or  $\mathcal{H}^{NS}$ . The two spaces are identified by  $\iota = \prod_{\ell>0} \sigma_\ell^x$  (in the naive picture), which is the disorder operator at the origin. The action of the VO's on  $\mathcal{H}_-$  is defined by  $\Phi^\sigma(\zeta)|_{\mathcal{H}_-} = \iota \circ \Phi^{-\sigma}(\zeta) \circ \iota$  and  $\Phi_\sigma(\zeta)|_{\mathcal{H}_-} = \iota \circ \Phi_{-\sigma}(\zeta) \circ \iota$ .

A priori there are  $2 \times 2$  possible choices of the boundary conditions for the full space on which the row-to-row Hamiltonian acts. For simplicity we shall fix the boundary condition for the left half to be  $+$ , and set  $\mathcal{F}_{+,\pm} = \text{Hom}(\mathcal{H}_\pm, \mathcal{H}_+) \simeq \mathcal{H}_+ \otimes \mathcal{H}_\pm^*$ . The dual space is  $\mathcal{F}_{+,\pm}^* = \text{Hom}(\mathcal{H}_+, \mathcal{H}_\pm) \simeq \mathcal{H}_+^* \otimes \mathcal{H}_\pm$  where the coupling between  $\mathcal{F}_{+,\pm}$  and  $\mathcal{F}_{+,\pm}^*$  is chosen to be  $\langle f|g \rangle = \text{tr}_{\mathcal{H}_\pm}(fg)$  for  $f \in \mathcal{F}_{+,\pm}$  and  $g \in \mathcal{F}_{+,\pm}^*$ . In this picture the ground state vector and its dual are  $|\text{vac}\rangle = \langle \text{vac}| = x^D$ .

The first way to introduce the creation and annihilation operators is via the explicit diagonalization of the Hamiltonian using fermions. Define

$$\varphi(z) = \sum_m \frac{1}{x^{2m} + x^{-2m}} \frac{\phi_m}{z^m}.$$

We find that the creation and annihilation operators  $\hat{\varphi}(z), \hat{\varphi}^*(z) : \mathcal{F}_{\pm,+} \rightarrow \mathcal{F}_{\mp,+}$  are given by

$$\begin{aligned}\hat{\varphi}(z)(v) &= \varphi(xz) \circ v \circ \iota - v \circ \varphi(z/x) \circ \iota, \\ \hat{\varphi}^*(z)(v) &= \varphi(z/x) \circ v \circ \iota + v \circ \varphi(xz) \circ \iota,\end{aligned}\tag{87}$$

where  $v \in \mathcal{F}_{\pm,+}$ . They satisfy

$$\hat{\varphi}(z)(|\text{vac}\rangle) = 0, \quad [\hat{\varphi}(z_1), \hat{\varphi}^*(z_2)]_+ = \delta(z_1/z_2).$$

The second way is to utilize the transfer matrices  $\tau_{NS}^R(\zeta), \tau_R^{NS}(\zeta)$  in the representation theoretical picture, as is done for the XXZ Hamiltonian and

other models [9],[13], [16]. Let us set

$$\begin{aligned}\tau_{NS}^R(\zeta)(v) &= (g^R)^{-1} \sum_{\sigma} \Phi(\zeta)_{\sigma} \circ v \circ \Phi(\zeta)^{\sigma}, \\ \tau_R^{NS}(\zeta)(v) &= (g^{NS})^{-1} \sum_{\sigma} \Phi(\zeta)^{\sigma} \circ v \circ \Phi(\zeta)_{\sigma}.\end{aligned}$$

If we define

$$\hat{\phi}(z)(v) = \phi(z/x) \circ v \circ \iota, \quad \hat{\phi}^*(z)(v) = \phi(xz) \circ v \circ \iota$$

then we have the following commutation relations:

$$\begin{aligned}\tau_{NS}^R(\zeta) \hat{\phi}^{NS}(z) &= f(z\zeta^2/x^2) \hat{\phi}^R(z) \tau_{NS}^R(\zeta), \\ \tau_R^{NS}(\zeta) \hat{\phi}^R(z) &= f(z\zeta^2/x^2) \hat{\phi}^{NS}(z) \tau_R^{NS}(\zeta).\end{aligned}$$

Here  $f(z)$  signifies the function defined in Appendix A, (88).

The two types operators defined above are related, for example, as follows.

$$\begin{aligned}\hat{\varphi}^*(z)(|\text{vac}\rangle) &= \hat{\phi}^*(z)(|\text{vac}\rangle) \\ \hat{\varphi}^*(z_1)(\hat{\varphi}^*(z_2)(|\text{vac}\rangle)) &= \hat{\phi}^*(z_1)(\hat{\phi}^*(z_2)(|\text{vac}\rangle)) - \delta(x^2 z_1/z_2)|\text{vac}\rangle.\end{aligned}$$

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## A Matrix elements of the Ising VO

In this appendix, we shall outline the derivation of the matrix elements of VO's.

Let us consider  $\Phi_{\text{R}}^{\text{NS},\text{V}}(1)$ . Define  $f(z)$  and  $c_r$  by

$$f(z) = -\sqrt{k}\text{sn}(v) = -ix^{1/2}z^{-1/2} \frac{(z; x^4)_{\infty}(x^4 z^{-1}; x^4)_{\infty}}{(x^2 z; x^4)_{\infty}(x^2 z^{-1}; x^4)_{\infty}}, \quad (88)$$

where  $v$  is related to  $z$  by  $z = \exp(\pi i v/I)$ . Using the Fourier expansion  $f(z) = \sum_{r \in \frac{1}{2} + \mathbf{Z}} c_r z^{-r}$  on  $|z| = 1$  and the formulas (57), (58) it can be shown that

$$\sum_{j=1}^{\infty} a_{rj}^{\text{NS}} a_{-sj+1}^{\text{R}} 2\eta_s^{-1} = c_{r-s}.$$

With the aid of this we rewrite the intertwining property (63) as follows.

$$\phi_r^{\text{NS}} \Phi_R^{\text{NS,V}}(1) = \sum_{s \in \mathbf{Z}} c_{r-s} \Phi_R^{\text{NS,V}}(1) \phi_s^{\text{R}}.$$

In terms of the generating functions

$$\phi^{\text{NS}}(z) = \sum_{r \in \mathbf{Z} + \frac{1}{2}} \phi_r^{\text{NS}} z^{-r}, \quad (89)$$

$$\phi^{\text{R}}(z) = \sum_{s \in \mathbf{Z}} \phi_s^{\text{R}} z^{-s}. \quad (90)$$

we have  $\phi^{\text{NS}}(z) \Phi_R^{\text{NS,V}}(1) = f(z) \Phi_R^{\text{NS,V}}(1) \phi^{\text{R}}(z)$ , or equivalently

$$\tilde{\phi}^{\text{NS}}(z) \Phi_R^{\text{NS,V}}(1) = \Phi_R^{\text{NS,V}}(1) \tilde{\phi}^{\text{R}}(z). \quad (91)$$

Here we have introduced auxiliary operators

$$\begin{aligned} \tilde{\phi}^{\text{NS}}(z) &= f_-(z) \phi^{\text{NS}}(z) = \sum_{n \in \mathbf{Z}} \tilde{\phi}_n^{\text{NS}} z^{-n}, \\ \tilde{\phi}^{\text{R}}(z) &= f_+(z) \phi^{\text{R}}(z) = \sum_{n \in \mathbf{Z}} \tilde{\phi}_n^{\text{R}} z^{-n} \end{aligned} \quad (92)$$

according to the factorization  $f(z) = f_+(z)/f_-(z)$ , with

$$f_+(z) = \frac{(z; x^4)_\infty}{(x^2 z; x^4)_\infty}, \quad f_-(z) = i x^{-1/2} z^{1/2} \frac{(x^2 z^{-1}; x^4)_\infty}{(x^4 z^{-1}; x^4)_\infty}.$$

Notice that we have

$$\begin{aligned} {}_{\text{NS}} \langle \text{vac} | \tilde{\phi}_n^{\text{NS}} &= 0 \quad (n < 0), \\ \tilde{\phi}_n^{\text{R}} | \text{vac} \rangle_{\text{R}} &= 0 \quad (n > 0), \quad \tilde{\phi}_0^{\text{R}} | \text{vac} \rangle_{\text{R}} = | \text{vac} \rangle_{\text{R}}. \end{aligned} \quad (93)$$

In view of (91) no confusion may arise if we abbreviate expressions like

$${}_{\text{NS}} \langle \text{vac} | \tilde{\phi}^{\text{NS}}(z) \Phi_R^{\text{NS,V}}(1) | \text{vac} \rangle_{\text{R}} = {}_{\text{NS}} \langle \text{vac} | \Phi_R^{\text{NS,V}}(1) \tilde{\phi}^{\text{R}}(z) | \text{vac} \rangle_{\text{R}}$$

to  ${}_{\text{NS}} \langle \text{vac} | \tilde{\phi}(z) | \text{vac} \rangle_{\text{R}}$  and so on. From the above properties, it follows that

$${}_{\text{NS}} \langle \text{vac} | \tilde{\phi}(z) \tilde{\phi}(w) | \text{vac} \rangle_{\text{R}} = \frac{(z-w)(z+w-(1+x^{-2})zw)}{(z-x^2w)(z-x^{-2}w)}.$$

Note that this formula contains  ${}_{\text{NS}} \langle \text{vac} | \tilde{\phi}(z) | \text{vac} \rangle_{\text{R}} = 1$  as a special case  $w = 0$ . The general matrix elements are given by Wick's theorem.

The other case  $\Phi_{\text{NS}}^{\text{R,V}}(1)$  can be treated quite similarly. The intertwining property becomes

$$\phi^{\text{R}}(z)\Phi_{\text{NS}}^{\text{R,V}}(1) = f(z)\Phi_{\text{NS}}^{\text{R,V}}(1)\phi^{\text{NS}}(z)$$

where  $f(z)$  is the same as (88). Defining auxiliary operators

$$\tilde{\phi}^{\text{R}'}(z) = f'_-(z)\phi^{\text{R}}(z), \quad \tilde{\phi}^{\text{NS}'}(z) = f'_+(z)\phi^{\text{NS}}(z)$$

with

$$f'_+(z) = -ix^{1/2}z^{-1/2}\frac{(z; x^4)_\infty}{(x^2z; x^4)_\infty}, \quad f'_-(z) = \frac{(x^2z^{-1}; x^4)_\infty}{(x^4z^{-1}; x^4)_\infty}$$

we have

$$\begin{aligned} \tilde{\phi}^{\text{R}'}(z)\Phi_{\text{NS}}^{\text{R,V}}(1) &= \Phi_{\text{NS}}^{\text{R,V}}(1)\tilde{\phi}^{\text{NS}'}(z), \\ {}_{\text{R}}\langle \text{vac} | \tilde{\phi}_n^{\text{R}'} &= 0 \ (n < 0), \quad {}_{\text{R}}\langle \text{vac} | \tilde{\phi}_0^{\text{R}'} = {}_{\text{R}}\langle \text{vac} |, \quad \tilde{\phi}_n^{\text{NS}'} | \text{vac} \rangle_{\text{NS}} = 0 \ (n > 0). \end{aligned}$$

We find

$${}_{\text{R}}\langle \text{vac} | \tilde{\phi}'(z)\tilde{\phi}'(w) | \text{vac} \rangle_{\text{NS}} = \frac{(z-w)(z+w-1-x^2)}{(z-x^2w)(z-x^{-2}w)}.$$

The matrix elements relative to the creation/annihilation operators in each sector can be derived from the above generating functions. Let  $X_{mn}$  and  $\gamma_m$  be as in (67). Using the formula

$$\frac{(x^2z; x^4)_\infty}{(z; x^4)_\infty} = \sum_{n=0}^{\infty} \gamma_n z^n$$

we obtain

$$\begin{aligned} {}_{\text{NS}}\langle \text{vac} | \Phi_{\text{R}}^{\text{NS,V}}(1)\phi_{-m}^{\text{R}}\phi_{-n}^{\text{R}} | \text{vac} \rangle_{\text{R}} &= X_{n,m}\gamma_m\gamma_n \quad (m, n \geq 0, m+n > 0) \\ {}_{\text{NS}}\langle \text{vac} | \phi_{m+1/2}^{\text{NS}}\Phi_{\text{R}}^{\text{NS,V}}(1)\phi_{-n}^{\text{R}} | \text{vac} \rangle_{\text{R}} &= X_{-m-1/2,n}ix^{2m+1/2}\gamma_m\gamma_n \quad (m, n \geq 0) \\ {}_{\text{NS}}\langle \text{vac} | \phi_{m+1/2}^{\text{NS}}\phi_{n+1/2}^{\text{NS}}\Phi_{\text{R}}^{\text{NS,V}}(1) | \text{vac} \rangle_{\text{R}} &= X_{m+1/2,n+1/2}x^{2m+2n+1}\gamma_m\gamma_n. \quad (m, n \geq 0) \end{aligned}$$

Likewise we have

$$\begin{aligned} {}_{\text{R}}\langle \text{vac} | \phi_m^{\text{R}}\phi_n^{\text{R}}\Phi_{\text{NS}}^{\text{R,V}}(1) | \text{vac} \rangle_{\text{NS}} &= X_{m,n}x^{2m+2n}\gamma_m\gamma_n \quad (m, n \geq 0, m+n > 0) \\ {}_{\text{R}}\langle \text{vac} | \phi_m^{\text{R}}\Phi_{\text{NS}}^{\text{R,V}}(1)\phi_{-n-1/2}^{\text{NS}} | \text{vac} \rangle_{\text{NS}} &= X_{m,-n-1/2}ix^{2m-1/2}\gamma_m\gamma_n \quad (m, n \geq 0) \\ {}_{\text{R}}\langle \text{vac} | \Phi_{\text{NS}}^{\text{R,V}}(1)\phi_{-m-1/2}^{\text{NS}}\phi_{-n-1/2}^{\text{NS}} | \text{vac} \rangle_{\text{NS}} &= X_{m+1/2,n+1/2}(-x^{-1})\gamma_m\gamma_n. \quad (m, n \geq 0) \end{aligned}$$

Finally, by making use of the identity ( $N$  even)

$$\text{Pfaffian} \left( \frac{x_i - x_j}{1 - x_i x_j} \right)_{1 \leq i, j \leq N} = \prod_{1 \leq i < j \leq N} \frac{x_i - x_j}{1 - x_i x_j}$$

we arrive at the general expressions (65), (66).

## B Spin correlation functions

The spin correlation functions for the Ising model are given as traces of VO's. They can be calculated using the intertwining property and the unitarity of the VO's alone, without referring to the formulas for the matrix elements. In this appendix, we illustrate this by the simplest case of the nearest diagonal order-order and disorder-disorder correlations.

Let us introduce some notations. Setting  $z = \exp(\pi iv/I)$  we define

$$f_0^{\text{NS}}(z) = \sqrt{\frac{2Ik}{\pi}} \text{cn}(v) = \sqrt{\frac{2\pi}{kI}} \sum_{r \in \mathbf{Z} + \frac{1}{2}} \eta_r^{-1} z^r,$$

$$f_0^{\text{R}}(z) = \sqrt{\frac{2I}{\pi}} \text{dn}(v) = \sqrt{\frac{2\pi}{I}} \sum_{s \in \mathbf{Z}} \eta_s^{-1} z^s.$$

They are related to  $f(z) = -\sqrt{k} \text{sn}(v)$  given in (88) by

$$f_0^{\text{NS}}(x^2 z) f(z) = i f_0^{\text{R}}(z), \quad f_0^{\text{R}}(x^{-2} z) f(z) = -i f_0^{\text{NS}}(z).$$

Define further

$$\psi_1^{\text{NS}}(\zeta) = \zeta^{-D^{\text{NS}}} \psi_1^{\text{NS}} \zeta^{D^{\text{NS}}} = \oint \frac{dz}{2\pi i z} f_0^{\text{NS}}(z) \phi^{\text{NS}}(z/\zeta^2),$$

$$\psi_1^{\text{R}}(\zeta) = \zeta^{-D^{\text{R}}} \psi_1^{\text{R}} \zeta^{D^{\text{R}}} = \oint \frac{dz}{2\pi i z} f_0^{\text{R}}(z) \phi^{\text{R}}(z/\zeta^2).$$

In terms of them and the generating functions (89), (90), the intertwining properties of the VO's are summarized as follows:

$$\begin{aligned} \phi^{\text{NS}}(z) \Phi^\sigma(\zeta) &= f(z\zeta^2) \Phi^{-\sigma}(\zeta) \phi^{\text{R}}(z), \\ \phi^{\text{R}}(z) \Phi_\sigma(\zeta) &= f(z\zeta^2) \Phi_{-\sigma}(\zeta) \phi^{\text{NS}}(z), \\ \sigma \Phi^\sigma(\zeta) &= \Phi^\sigma(\zeta) \psi_1^{\text{R}}(\zeta), \\ \sigma \Phi_\sigma(\zeta) &= -i \Phi_{-\sigma}(\zeta) \psi_1^{\text{NS}}(\zeta). \end{aligned}$$

Rewriting the properties for the horizontal VO's we have further

$$\begin{aligned} \Phi^\sigma(x\zeta) \phi^{\text{R}}(z) &= f(z\zeta^2) \phi^{\text{NS}}(z) \Phi^{-\sigma}(x\zeta), \\ \Phi_\sigma(x\zeta) \phi^{\text{NS}}(z) &= f(z\zeta^2) \phi^{\text{R}}(z) \Phi_{-\sigma}(x\zeta), \\ \sigma \Phi^\sigma(x\zeta) &= i \psi_1^{\text{NS}}(\zeta) \Phi^{-\sigma}(\zeta), \\ \sigma \Phi_\sigma(x\zeta) &= \psi_1^{\text{R}}(\zeta) \Phi_\sigma(\zeta). \end{aligned}$$

The order-order and disorder-disorder correlators for the nearest diagonal neighbors are given respectively by

$$\frac{\sum_{\sigma\sigma'} \sigma\sigma' \text{tr}_{\mathcal{H}^{\text{NS}}} \left( x^{2D^{\text{NS}}} \Phi^{\sigma'}(x\zeta_2) \Phi_\sigma(x\zeta_1) \Phi^\sigma(\zeta_1) \Phi_{\sigma'}(\zeta_2) \right)}{g^{\text{NS}} g^{\text{R}} \text{tr}_{\mathcal{H}^{\text{NS}}} (x^{2D^{\text{NS}}})}, \quad (94)$$

$$\frac{\sum_{\sigma} \text{tr}_{\mathcal{H}^R} \left( x^{2D^R} \Phi_{\sigma}(x\zeta_2) \Phi^{\sigma}(x\zeta_1) \Phi_{-\sigma}(\zeta_1) \Phi^{-\sigma}(\zeta_2) \right)}{g^{\text{NS}} g^R \text{tr}_{\mathcal{H}^R} (x^{2D^R})}. \quad (95)$$

Here the  $\zeta_i$  are the spectral parameters attached to the intermediate lines as shown below.

Consider (94). Using the intertwining properties and the unitarity of the VO's we have

$$\sum_{\sigma} \sigma \Phi_{\sigma}(x\zeta_1) \Phi^{\sigma}(\zeta_1) = \sum_{\sigma} \Phi_{\sigma}(x\zeta_1) \Phi^{\sigma}(\zeta_1) \psi^R(\zeta_1)_1 = g^R \psi^R(\zeta_1)_1.$$

Hence, setting  $\zeta = \zeta_1/\zeta_2$ , we find

$$\begin{aligned} & \sum_{\sigma\sigma'} \sigma\sigma' \Phi^{\sigma'}(x\zeta_2) \Phi_{\sigma}(x\zeta_1) \Phi^{\sigma}(\zeta_1) \Phi_{\sigma'}(\zeta_2) \\ &= g^R \sum_{\sigma'} \sigma' \Phi^{\sigma'}(x\zeta_2) \psi^R(\zeta_1)_1 \Phi_{\sigma'}(\zeta_2) \\ &= g^R \oint \frac{dz}{2\pi iz} f_0^R(z) \sum_{\sigma'} \sigma' \Phi^{\sigma'}(x\zeta_2) \phi^R(z/\zeta_1^2) \Phi_{\sigma'}(\zeta_2) \\ &= g^R \oint \frac{dz}{2\pi iz} f_0^R(z) f(z/\zeta^2) \phi^{\text{NS}}(z/\zeta_1^2) \sum_{\sigma'} \sigma' \Phi^{-\sigma'}(x\zeta_2) \Phi_{\sigma'}(\zeta_2) \\ &= -i g^R g^{\text{NS}} \oint \frac{dz}{2\pi iz} f_0^R(z) f(z/\zeta^2) \phi^{\text{NS}}(z/\zeta_1^2) \psi_1^{\text{NS}}(\zeta_2). \end{aligned}$$

To take the trace we invoke the following simple lemma:

$$\begin{aligned} \frac{\text{tr}_{\mathcal{H}^R} \left( x^{2D^R} \phi^R(z_1) \phi^R(z_2) \right)}{\text{tr}_{\mathcal{H}^R} (x^{2D^R})} &= \delta^R(x^2 z_1/z_2), \quad \delta^R(z) = \sum_{r \in \mathbf{Z}} z^r, \\ \frac{\text{tr}_{\mathcal{H}^{\text{NS}}} \left( x^{2D^{\text{NS}}} \phi^{\text{NS}}(z_1) \phi^{\text{NS}}(z_2) \right)}{\text{tr}_{\mathcal{H}^{\text{NS}}} (x^{2D^{\text{NS}}})} &= \delta^{\text{NS}}(x^2 z_1/z_2), \quad \delta^{\text{NS}}(z) = \sum_{r \in \frac{1}{2} + \mathbf{Z}} z^r. \end{aligned}$$

Here the  $\delta^{\text{NS}}(z)$  contains half odd integer powers in  $z$ , but in the course of the computation only integer powers appear. With the aid of these formulas, (94) becomes

$$\oint \frac{dz}{2\pi iz} f_0^R(z) f_0^R(z/\zeta^2) = \frac{2\pi}{I} \sum_{s \in \mathbf{Z}} \eta_s^{-2} \zeta^{2s}.$$

The case of (95) can be treated similarly. We find that it is given by

$$\oint \frac{dz}{2\pi iz} f_0^{\text{NS}}(z) f_0^{\text{NS}}(z/\zeta^2) = \frac{2\pi}{kI} \sum_{r \in \mathbf{Z} + \frac{1}{2}} \eta_r^{-2} \zeta^{2r}.$$

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